Bost-Connes type systems for function fields

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Algebraic Geometry

Let X be a compact topological space. There is a **bijection :**

$$\textbf{points of } X \leftrightarrow \textbf{maximal ideals of } C(X)$$

For $f \in C(X)$, the value of f at I is the class of f in the **residue field** $C(X)/I = \mathbb{C}$.

Let A be a commutative unital ring.
Definition : $\mathsf{Spec}\, A$ is the set of all prime ideals of A.

For $I \in \mathsf{Spec}\, A$, the **residue field** of I is the field of fractions of A/I.

$$V(J) = \{I \in \mathsf{Spec}\, A \mid I \supset J\}.$$

Zariski topology : the closed subsets are the $V(J)$ for all ideals J of A.

Example : $A = \mathbb{Z}$

$$\operatorname{Spec}\mathbb{Z} \;=\; \{(0),\,(2),\,(3),\,(5),\,(7),\,(11),\,\ldots\}.$$

Residue fields :

$$\mathbb{Q},\ \mathbb{F}_2,\ \mathbb{F}_3,\ \mathbb{F}_5,\ \mathbb{F}_7,\ \mathbb{F}_{11},\ \ldots$$

Riemann zeta function :

$$\zeta(s) \;=\; \prod_p \frac{1}{1-p^{-s}}.$$

Example : $A = \mathbb{F}_2[X]$

$\operatorname{Spec} \mathbb{F}_2[X] = \{(0), (X), (X+1),$
$(X^2+X+1), (X^3+X+1),$
$(X^3+X^2+1), \ldots\}$

Residue fields :

$$\mathbb{F}_2(X), \mathbb{F}_2, \mathbb{F}_2, \mathbb{F}_4, \mathbb{F}_8, \mathbb{F}_8 \ \ldots$$

$\mathbb{F}_2(X)$ is an example of a **function field**.

Zeta function :

$$\zeta_A(s) \ = \ \prod_{\mathfrak{p}} \frac{1}{1 - \mathbf{N}\mathfrak{p}^{-s}}.$$

The **norm** $\mathbf{N}\mathfrak{p}$ of a closed point $\mathfrak{p}$ is the cardinal of the residue field $A/\mathfrak{p}$.

Function fields

Definition : A function field is a finite extension of $\mathbb{F}_p(X)$, for some prime number p.

The **constant subfield** of k is the largest finite field included in k. It is isomorphic to $\mathbb{F}_q$ with $q = p^n$ for some $n \geqslant 1$.

Let A be the integral closure of $\mathbb{F}_p[X]$ in k.

Example : $k = \mathbb{F}_q(X)$, the constant subfield is $\mathbb{F}_q$, and $A = \mathbb{F}_q[X]$.

The curve Spec A

$$\mathrm{Spec}\, A \;=\; \{(0)\} \;\cup\; \{\,\text{maximal ideals } \mathfrak{p} \text{ of } A\,\}.$$

The residue field at (0) is k.
The residue field at $\mathfrak{p}$, noted $\mathbb{F}_{\mathfrak{p}}$, is isomorphic to $\mathbb{F}_{q^{n_{\mathfrak{p}}}}$ for some $n_{\mathfrak{p}} \geqslant 1$
The norm of $\mathfrak{p}$ is $\mathbf{N}\mathfrak{p} = q^{n_{\mathfrak{p}}}$.

$$\zeta_A(s) \;=\; \prod_{\mathfrak{p}} \frac{1}{1 - \mathbf{N}\mathfrak{p}^{-s}}.$$

Theorem (A. Weil, 1940) : The Riemann Hypothesis holds for ζ_A.

Consequence : the number of $\mathfrak{p}$ with $\mathbf{N}\mathfrak{p} = q^n$ is

$$\frac{q^n}{n} \;+\; O(q^{n/2}).$$

The Bost-Connes system

$(C_{\mathbb{Z}}, \sigma_t)$ is a C^*-dynamical system associated to $\operatorname{Spec} \mathbb{Z}$.

Definition : $C_{\mathbb{Z}} = C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N}^*$.
(notation : $\mathbb{N}^* = \mathbb{Z}_{>0}$ multiplicative semigroup)

Here, $\mathbb{N}^*$ acts on $C^*(\mathbb{Q}/\mathbb{Z})$ by :

$$\forall \gamma \in \mathbb{Q}/\mathbb{Z}, \quad n \cdot e(\gamma) = \frac{1}{n} \sum_{n\delta = \gamma} e(\delta).$$

Thus, $C_{\mathbb{Z}}$ is generated over $C^*(\mathbb{Q}/\mathbb{Z})$ by elements μ_n $(n \in \mathbb{N}^*)$ satisfying

$$\forall \gamma \in \mathbb{Q}/\mathbb{Z}, \quad \mu_n \, e(\gamma) \, \mu_n^* = \frac{1}{n} \sum_{n\delta = \gamma} e(\delta).$$

Flow on $C_{\mathbb{Z}}$: $\sigma_t(\mu_n) = n^{it} \mu_n$
and the subalgebra $C^*(\mathbb{Q}/\mathbb{Z})$ is fixed by the flow σ_t.

Classification of KMS$_\beta$ states of $(C_{\mathbb{Z}}, \sigma_t)$

For $0 < \beta \leqslant 1$ (**high temperature**), there is a **unique** KMS$_\beta$ state φ_β.

The corresponding factor is the hyperfinite type III$_1$ factor.

For $1 < \beta < \infty$ (**low temperature**), the **extremal** KMS$_\beta$ states are type I$_\infty$ Gibbs states :

$$\varphi_{\beta,\pi}(x) \;=\; \frac{\mathrm{Tr}(e^{-\beta H}\,\pi(x))}{\mathrm{Tr}(e^{-\beta H})}.$$

parametrized by irreducible representations π of $C_{\mathbb{Z}}$.

The space of **extremal** KMS$_\beta$ states is **principal homogeneous** under $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$.

$\mathrm{Tr}(e^{-\beta H}) = \zeta(\beta)$ Riemann zeta function.

Bost-Connes for function fields

By what to replace $\mathbb{N}^*$ and $\mathbb{Q}/\mathbb{Z}$?

$\mathbb{N}^* \quad \longrightarrow \quad$ ideals of A. Prime numbers are replaced by maximal ideals of A, *i.e.* closed points of $\mathrm{Spec}\, A$.

$\mathbb{Q}/\mathbb{Z} \quad \longrightarrow \quad$ torsion points of Drinfel'd modules.

Idea : the formula for $\sin(z)$

$$\sin z \;=\; z \prod_{t \in 2\pi\mathbb{Z} - \{0\}} \left(1 - \frac{z}{t}\right) \cdot$$

still makes sense in positive characteristic.
$\longrightarrow$ notion of Drinfel'd module.

Allows to **construct** a C^*-dynamical system

$$(C_A,\ \sigma_t).$$

Low-temperature KMS$_\beta$ states of (C_A, σ_t)

$1 < \beta < \infty$

The **extremal** KMS$_\beta$ states are type I$_\infty$ Gibbs states :

$$\varphi_{\beta,\pi}(x) \;=\; \frac{\mathrm{Tr}(e^{-\beta H}\,\pi(x))}{\mathrm{Tr}(e^{-\beta H})}.$$

parametrized by irreducible representations π of C_A.

The space of **extremal** KMS$_\beta$ states is **principal homogeneous** under $\mathrm{Gal}(K/k)$ with

$$k^{\mathrm{ab},\infty} \subset K \subset k^{\mathrm{ab}}.$$

$\mathrm{Tr}(e^{-\beta H}) = \zeta_A(\beta)$ zeta function.

Pole at $\beta = 1 \;\; \rightarrow$ **phase transition**.

Unicity of high-temp. KMS$_\beta$ state ($0 < \beta \leqslant 1$)

There is a **unique** KMS$_\beta$ state φ_β.

Idea (BC) : decompose C_A into spectral subspaces.

symmetry group $G = \mathrm{Gal}(K/k)$ acts on C_A.

G is abelian, profinite. Hence $\widehat{G}$ is abelian, discrete.

For any $\chi \in \widehat{G}$, **spectral subspace**

$$C_{A,\chi} \;=\; \{x \in C_A \;\mid\; \forall g \in G,\; gx = \chi(g)x\}.$$

$$C_A \quad = \quad \text{norm-closure of} \quad \bigoplus_{\chi \in \widehat{G}} C_{A,\chi}.$$

Type of high-temp. KMS$_\beta$ state ($0 < \beta \leqslant 1$)

Let M (resp. M_1) be the **weak closure** of C_A (resp. $C_{A,1}$) in the GNS representation of φ_β.

M is a **factor** because φ_β is the **unique** KMS$_\beta$ state.

M_1 is an **ITPFI** :

$$M_1 = \bigotimes_{\mathfrak{p}} \left(M_{1,\mathfrak{p}},\ \varphi_{\beta,\mathfrak{p}} \right).$$

M_1 is the hyperfinite type III$_{q^{-\beta}}$ factor.

$\longrightarrow$ the centralizer M_{1,φ_β} is a **factor** (type II$_1$).

Difficult part : deduce that the centralizer M_{φ_β} is **also a factor** (type II$_1$).

One can then prove that :
M **is the hyperfinite type III$_{q^{-\beta}}$ factor.**