Re-engineering the Mathematical Foundations of Supersymmetry With Adinkras & Clifford Algebras

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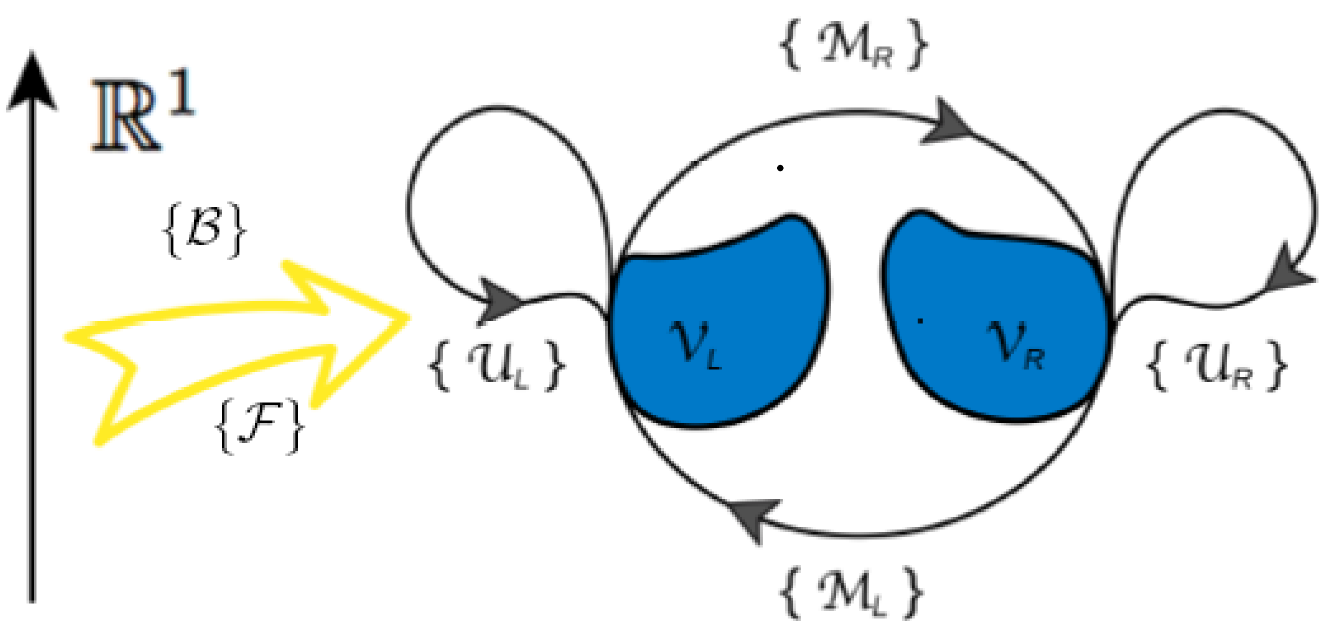
A Proposal for a Coordinate Independent Definition of 1D Supersymmetry



Let $\mathbb{R}^1$ represent the real line, $\{\mathcal{B}\}$ a set of commuting maps, and $\{\mathcal{F}\}$ a set of anti-commuting maps ($\mathcal{B} \in \{\mathcal{B}\}$ and $\mathcal{F} \in \{\mathcal{F}\}$)

$$\mathcal{B} : \mathbb{R}^1 \rightarrow \mathbb{D}_{(1)}\Big[\mathcal{GR}(\mathrm{d}, \mathcal{N})\Big]$$

$$\mathcal{F} : \mathbb{R}^1 \rightarrow \mathbb{D}_{(2)}\Big[\mathcal{GR}(\mathrm{d}, \mathcal{N})\Big]$$

where $\mathbb{D}_{(1)}$ and $\mathbb{D}_{(2)}$ are reps of $\mathcal{GR}(\mathrm{d}, \mathcal{N})$

Definition of $\mathcal{GR}$(d, $\mathcal{N}$)

$\mathcal{V}_L$ and $\mathcal{V}_R$ real d-dimensional vector spaces equipped with Euclidean inner product structures $\phi \in \mathcal{V}_L \qquad \psi \in \mathcal{V}_R$

$\{\mathcal{M}_L\}$, $\{\mathcal{M}_R\}$, $\{\mathcal{U}_L\}$, $\{\mathcal{U}_R\}$, linear maps

$\mathcal{M}_L \in \{\mathcal{M}_L\}, \mathcal{M}_R \in \{\mathcal{M}_L\}, \mathcal{U}_L \in \{\mathcal{U}_L\},$
$\mathcal{U}_R \in \{\mathcal{U}_R\}$

$$\mathcal{M}_L : \mathcal{V}_R \;\rightarrow\; \mathcal{V}_L \quad , \quad \mathcal{M}_R : \mathcal{V}_L \;\rightarrow\; \mathcal{V}_R \quad ,$$
$$\mathcal{U}_L : \mathcal{V}_L \;\rightarrow\; \mathcal{V}_L \quad , \quad \mathcal{U}_R : \mathcal{V}_R \;\rightarrow\; \mathcal{V}_R \qquad .$$

$\dim\{\mathcal{M}_L\} = \dim\{\mathcal{M}_R\} = \dim\{\mathcal{U}_L\} = \dim\{\mathcal{U}_R\}$
$= \mathrm{d}^2$.

$$\mathcal{M}_R \circ \mathcal{M}_L : \mathcal{V}_R \;\rightarrow\; \mathcal{V}_R \quad ,$$
$$\mathcal{M}_L \circ \mathcal{M}_R : \mathcal{V}_L \;\rightarrow\; \mathcal{V}_L \quad ,$$

These imply

$$\mathcal{M}_R \circ \mathcal{M}_L \;\in\; \mathcal{U}_R \quad ,$$
$$\mathcal{M}_L \circ \mathcal{M}_R \;\in\; \mathcal{U}_L \quad ,$$

$\{\mathrm{L}\} \in \{\mathcal{M}_L\}$ and $\{\mathrm{R}\} \in \{\mathcal{M}_R\}$,

$\mathrm{rank}(\{\mathrm{L}\}) = \mathrm{rank}(\{\mathrm{R}\}) = \mathcal{N} \leq d^2$

Let α and β be fixed vectors in $\mathbb{R}^{\mathcal{N}}$ then $\mathrm{L}(\alpha) \in \{\mathrm{L}\}$ and $\mathrm{R}(\beta) \in \{\mathrm{R}\}$ describe fixed elements in each subset.

We impose the requirements

$$\begin{aligned} \mathrm{L}(\alpha) \circ \mathrm{R}(\beta) + \mathrm{L}(\beta) \circ \mathrm{R}(\alpha) &= \\ -\, 2\,(\alpha,\, \beta)\, \mathrm{I}_{\mathcal{V}_L} &, \\ \mathrm{R}(\alpha) \circ \mathrm{L}(\beta) + \mathrm{R}(\beta) \circ \mathrm{L}(\alpha) &= \\ -\, 2\,(\alpha,\, \beta)\, \mathrm{I}_{\mathcal{V}_R} &, \end{aligned} \tag{1}$$

where $\mathrm{I}_{\mathcal{V}_L}$ and $\mathrm{I}_{\mathcal{V}_R}$ are the identity maps acting on the respective vector spaces $\mathcal{V}_L$ and $\mathcal{V}_R$ and $(\alpha,\, \beta)$ is the Euclidean inner product of α and β.

Finally, there is one other condition to be imposed. For all $\phi \in \mathcal{V}_L$ and $\psi \in \mathcal{V}_R$ we require

$$\langle \phi,\, \mathrm{L}(\alpha) : \psi \rangle \;=\; -\, \langle \mathrm{R}(\alpha) : \psi,\, \phi \rangle \quad , \tag{2}$$

where $\langle\ ,\ \rangle$ denotes the Euclidean inner product imposed both $\mathcal{V}_L$ and $\mathcal{V}_R$.

With a specific choice of coordinates, conditions (1) and (2) define a set of matrices that are generalized ($\mathcal{G}$) real ($\mathcal{R}$) versions of the Pauli matrices used in physics. Therefore, the algebraic structure defined by (1) and (2) has been given the name $\mathcal{GR}(\mathrm{d}, \mathcal{N})$.

Ubiquitous Realization of SUSY

Two maps such that $\dim(\mathcal{B}) = \dim(\mathcal{F})$ form a representation of supersymmetry if;

$$\delta_Q(\epsilon) \circ \mathcal{B} \equiv i\,\mathrm{L}(\epsilon) \circ \mathcal{F} \quad ,$$

$$\delta_Q(\epsilon) \circ \mathcal{F} \equiv \mathrm{R}(\epsilon) \circ \frac{d\mathcal{B}}{dt} \quad ,$$

(where ϵ is an element of an anticommuting algebra) then implies

$$\Big[\, \delta_Q(\epsilon_1)\,,\, \delta_Q(\epsilon_2) \Big] \circ \mathcal{B} = - i2\,(\epsilon_1,\, \epsilon_2)\, \frac{d\mathcal{B}}{dt} \quad ,$$

$$\Big[\, \delta_Q(\epsilon_1)\,, \delta_Q(\epsilon_2) \Big] \circ \mathcal{F} = - i2\,(\epsilon_1,\, \epsilon_2)\, \frac{d\mathcal{F}}{dt} \quad ,$$

for two such anticommuting elements ϵ_1 and ϵ_2 and

$$\delta_Q(\epsilon_1) \circ \delta_Q(\epsilon_2) - \delta_Q(\epsilon_2) \circ \delta_Q(\epsilon_1) = \Big[\, \delta_Q(\epsilon_1)\,,\, \delta_Q(\epsilon_2) \Big] \quad ,$$

Example (1)

$\{\mathcal{B}\} : \mathbb{R}^1 \to \mathcal{V}_L$ and $\{\mathcal{F}\} : \mathbb{R}^1 \to \mathcal{V}_R$.

In this case, there are introduced functions $\phi_i(\tau)$ for $\{\mathcal{B}\}$ and $\psi_{\hat{k}}(\tau)$ for $\{\mathcal{F}\}$ along with equations

$$
\begin{aligned}
\mathrm{L}_K \, \mathrm{R}_P + \mathrm{L}_P \, \mathrm{R}_K &= -\, 2 \, \delta_{KP} \, \mathrm{I} \quad , \\
\mathrm{R}_K \, \mathrm{L}_P + \mathrm{R}_P \, \mathrm{L}_K &= -\, 2 \, \delta_{KP} \, \mathrm{I} \quad .
\end{aligned}
$$

$$
\begin{aligned}
\delta_Q \, \phi_i &= i \, \epsilon^K \, (\mathrm{L}_K)_i{}^{\hat{j}} \, \psi_{\hat{j}} \quad , \\
\delta_Q \, \psi_{\hat{i}} &= -\, \epsilon^K \, (\mathrm{R}_K)_{\hat{i}}{}^j \, \partial_\tau \phi_j \quad ,
\end{aligned}
$$

Example (2)

$\{\mathcal{B}\} : \mathbb{R}^1 \to \mathcal{U}_L$ and $\{\mathcal{F}\} : \mathbb{R}^1 \to \mathcal{M}_R$.

$$\Phi_{k\,l} \in \{\mathcal{U}_L\} \quad , \quad \Psi_{\hat{k}\,l} \in \{\mathcal{M}_R\} \quad .$$

$$\begin{aligned} \delta_Q \Phi_{k\,l} &= \; i\,\epsilon^{\rm I}\,({\rm L}^{\rm I})_k{}^{\hat{\ell}}\,\Psi_{\hat{\ell}\,l} \quad , \\ \delta_Q \Psi_{\hat{k}\,l} &= \; -\,\epsilon^{\rm I}\,({\rm R}^{\rm I})_{\hat{k}}{}^{\ell}\,\partial_\tau \Phi_{\ell\,l} \quad , \end{aligned}$$

For a fixed value of $\mathcal{N}$ there is a minimum value $d_{\mathcal{N}}$ such that ${\rm d}_{\mathcal{N}} \times {\rm d}_{\mathcal{N}}$ matrices faithfully represent this algebra. This is illustrated by a following table.

We first write

$$\mathcal{N} \;=\; 8\,m \;+\; n$$

with

$$1 \;\le\; n \le\; 8$$

and use the definition $\mathcal{N} = 8\,k \rightarrow m = k$ - 1 for $k = 1,\, 2,\, \ldots\, \infty$

n	$F_{\mathcal{R}\,\mathcal{H}}(n)$
1	1
2	2
3	4
4	4
5	8
6	8
7	8
8	8

Table I

This is the Radon-Hurwitz function as noted by Pashnev & Toppan and can be used to write

$$\mathrm{d}_{\mathcal{N}} \;=\; 16^{m}\, F_{\mathcal{R}\,\mathcal{H}}(n)$$

Using A Matrix Representation to Define Basis Elements

Consider real matrices (with $I = 1, \ldots ,$ $\mathcal{N}+1$):

$$\gamma^I \gamma^J + \gamma^J \gamma^I = -2\eta^{IJ}\,\mathbf{I}$$

where $\eta^{IJ} = \text{diag}(1, \ldots, 1, -1)$.

Projection Operator

$$P_{\pm} = \tfrac{1}{2}(\mathbf{I} \pm \gamma^{\mathcal{N}+1})$$

L/R Definitions

$$L_{\rm I} \equiv P_+ \gamma_{\rm I} P_- \quad , \quad R_{\rm I} \equiv P_- \gamma_{\rm I} P_+ \quad ,$$

with $\mathrm{I} = 1, \ldots \mathcal{N}$

$$L_{\rm I} R_{\rm K} + L_{\rm K} R_{\rm I} = -2\delta_{\rm I\,K} P_+ \quad ,$$
$$R_{\rm I} L_{\rm K} + R_{\rm K} L_{\rm I} = -2\delta_{\rm I\,K} P_- \quad ,$$
$$\forall\, \mathrm{I}, \mathrm{K} = 1, \cdots, \mathcal{N} \quad .$$

For ordinary γ-matrices, it is customary in most of the physics literature to use a "wedge" product to complete the algebra of all γ-matrices to a covering algebra.

$$\{\Gamma\} = \{\, \mathbf{I}, \gamma^{I}, \gamma^{I_1} \wedge \gamma^{I_2}, \gamma^{I_1} \wedge \gamma^{I_2} \wedge \gamma^{I_3}, \\ \dots, \gamma^{I_1} \wedge \cdots \wedge \gamma^{I_{\mathcal{N}+1}} \,\}$$

Normal Type ($2^{\mathcal{N}} = (\mathrm{d}_{\mathcal{N}})^2$)

$$\{\Gamma\} = \{\, \mathbf{I}, \gamma^{I}, \gamma^{I_1} \wedge \gamma^{I_2}, \gamma^{I_1} \wedge \gamma^{I_2} \wedge \gamma^{I_3}, \\ \dots, \gamma^{I_1} \wedge \cdots \wedge \gamma^{I_{\mathcal{N}+1}} \,\} \oplus \\ \{\, \mathcal{D}, \gamma^{I} \mathcal{D}, \gamma^{I_1} \wedge \gamma^{I_2} \mathcal{D}, \\ \gamma^{I_1} \wedge \gamma^{I_2} \wedge \gamma^{I_3} \mathcal{D}, \\ \dots, \gamma^{I_1} \wedge \cdots \wedge \gamma^{I_{\mathcal{N}+1}} \mathcal{D} \,\}$$

Almost Complex Type ($2^{\mathcal{N}} = 2\,(\mathrm{d}_{\mathcal{N}})^2$)

$\mathcal{D}$ anti-commutes with others and satisfies $\mathcal{D}^2 = -\mathbf{I}$

$$
\begin{aligned}
\{\Gamma\} \ = \ \{ \, & \mathbf{I}, \gamma^{I}, \gamma^{I_1} \wedge \gamma^{I_2}, \gamma^{I_1} \wedge \gamma^{I_2} \wedge \gamma^{I_3}, \\
& \ldots, \gamma^{I_1} \wedge \cdots \wedge \gamma^{I_{\mathcal{N}+1}} \, \} \ \oplus \\
& \{ \, \mathcal{E}^{\hat{\alpha}}, \gamma^{I} \, \mathcal{E}^{\hat{\alpha}}, \gamma^{I_1} \wedge \gamma^{I_2} \mathcal{E}^{\hat{\alpha}}, \\
& \gamma^{I_1} \wedge \gamma^{I_2} \wedge \gamma^{I_3} \mathcal{E}^{\hat{\alpha}}, \\
& \ldots, \gamma^{I_1} \wedge \cdots \wedge \gamma^{I_{\mathcal{N}+1}} \, \mathcal{E}^{\hat{\alpha}} \, \}
\end{aligned}
$$

Quaternionic Type ($2^{\mathcal{N}} = 4 \, (\mathrm{d}_{\mathcal{N}})^2$)
$\mathcal{E}^{\hat{\alpha}}$ commutes with others and satisfies

$$
\mathcal{E}^{\hat{\alpha}} \, \mathcal{E}^{\hat{\beta}} \ = \ - \, \delta^{\hat{\alpha} \, \hat{\beta}} \, \mathbf{I} \ + \ \epsilon^{\hat{\alpha} \, \hat{\beta} \, \hat{\gamma}} \, \mathcal{E}^{\hat{\gamma}}
$$

Okubo

As one changes $\mathcal{N}$ the type of covering algebra changes according to the following table.

$\mathcal{GR}(\mathrm{d}_{\mathcal{N}}, \mathcal{N})$	$\mathcal{EGR}$ $(\mathrm{d}_{\mathcal{N}}, \mathcal{N})$ generators	Type
$\mathcal{GR}(8,8)$	f_{I}	N
$\mathcal{GR}(8,7)$	f_{I}	N
$\mathcal{GR}(8,6)$	f_{I}, $\mathcal{D}$	AC
$\mathcal{GR}(8,5)$	f_{I}, $\mathcal{E}^{\hat{\alpha}}$	Q
$\mathcal{GR}(4,4)$	f_{I}, $\mathcal{E}^{\hat{\alpha}}$	Q
$\mathcal{GR}(4,3)$	f_{I}, $\mathcal{E}^{\hat{\alpha}}$	Q
$\mathcal{GR}(2,2)$	f_{I}, $\mathcal{D}$	AC
$\mathcal{GR}(1,1)$	f	N

Table II

$$\mathcal{EGR} \equiv \{\mathcal{M}_L\} \oplus \{\mathcal{M}_R\} \oplus \{\mathcal{U}_L\} \oplus \{\mathcal{U}_R\}$$

For the $C(\mathcal{N}, 1)$ Clifford algebras we can always find a basis such that:

$$(\gamma^{\mathcal{N}+1})^T = \gamma^{\mathcal{N}+1}$$
$$(\gamma^{\mathrm{I}})^T = -\gamma^{\mathrm{I}}$$

then it follows that:

$$L_{\rm I}^T = (P_+ \gamma_{\rm I} P_-)^T = P_-^T \gamma_{\rm I}^T P_+^T$$
$$= -P_- \gamma_{\rm I} P_+ = -R_{\rm I}$$

Defining the $P_\pm$ as before leads to four classes of matrices that we denote by $\{\mathcal{U}_L\}$, $\{\mathcal{M}_L\}$, $\{\mathcal{U}_R\}$ and $\{\mathcal{M}_R\}$. In the normal case these are

$$\{\mathcal{U}_L\} = \{ P_+, P_+ \gamma_{\rm IJ} P_+ , ..., P_+ \gamma_{[\mathcal{N}]} P_+ \} \quad ,$$
$$\{\mathcal{M}_L\} = \{ P_+ \gamma^{\rm I} P_- , ..., P_+ \gamma_{[\mathcal{N}-1]} P_- \} \quad ,$$
$$\{\mathcal{U}_R\} = \{ P_-, P_- \gamma^{\rm IJ} P_- , ..., P_- \gamma_{[\mathcal{N}]} P_- \} \quad ,$$
$$\{\mathcal{M}_R\} = \{ P_- \gamma^{\rm I} P_+ , ..., P_- \gamma_{[\mathcal{N}-1]} P_+ \} \quad .$$

Obvious identifications of $\{\mathcal{U}_L\}$, $\{\mathcal{M}_L\}$, $\{\mathcal{U}_R\}$ and $\{\mathcal{M}_R\}$ can be made in the AC and Q cases.

Example (3)

$$\{\mathcal{F}\} : \mathbb{R}^1 \rightarrow \mathcal{M}_L \oplus \mathcal{M}_R$$

$$\{\mathcal{B}\} : \mathbb{R}^1 \rightarrow \mathcal{U}_L \oplus \mathcal{U}_R .$$

$$\delta_Q \Phi_{k\,l} = i \left[\epsilon^{\rm I} \, ({\rm L}^{\rm I})_k{}^{\hat{\ell}} \, \Psi_{\hat{\ell}\,l} \; + \; \bar{\epsilon}^{\rm I} \, ({\rm L}^{\rm I})_l{}^{\hat{\ell}} \, \widehat{\Psi}_{k\,\hat{\ell}} \right] \quad ,$$

$$\delta_Q \Psi_{\hat{k}\,l} = \left[- \epsilon^{\rm I} \, ({\rm R}^{\rm I})_{\hat{k}}{}^{\ell} \, \partial_\tau \Phi_{\ell\,l} \; + \; \bar{\epsilon}^{I} \, ({\rm L}^{\rm I})_l{}^{\hat{\ell}} \, \partial_\tau \widehat{\Phi}_{\hat{k}\,\hat{\ell}} \right]$$

$$\delta_Q \widehat{\Psi}_{k\,\hat{l}} = \left[- \epsilon^{\rm I} \, ({\rm L}^{\rm I})_k{}^{\hat{\ell}} \, \partial_\tau \widehat{\Phi}_{\hat{\ell}\,\hat{l}} \; - \; \bar{\epsilon}^{\rm I} \, ({\rm R}^{\rm I})_{\hat{l}}{}^{\ell} \, \partial_\tau \Phi_{k\,\ell} \right] \quad ;$$

$$\delta_Q \widehat{\Phi}_{\hat{k}\,\hat{l}} = i \left[\epsilon^{\rm I} \, ({\rm R}^{\rm I})_{\hat{k}}{}^{\ell} \, \widehat{\Psi}_{\ell\,\hat{l}} \; - \; \bar{\epsilon}^{I} \, ({\rm R}^{\rm I})_{\hat{l}}{}^{\ell} \, \Psi_{\hat{k}\,\ell} \right] \quad ,$$

Since $\mathcal{GR}(2,2)$ is an almost complex case its possible 'remove' elements in $\{\mathcal{M}_L\}$, $\{\mathcal{M}_R\}$, $\{\mathcal{U}_L\}$ and $\{\mathcal{U}_R\}$ that depend on the almost complex structure $\mathcal{D}$.

We introduce four τ-dependent "fields" $\hat{\Psi}_{k\hat{l}}$, $\Psi_{\hat{k}l}$, Φ_{kl} and $\hat{\Phi}_{\hat{k}\hat{l}}$ so that

$$\hat{\Psi}_{k\,\hat{l}} \in \{\mathcal{M}_L\}/\mathcal{D} \quad , \quad \Psi_{\hat{k}\,l} \in \{\mathcal{M}_R\}/\mathcal{D} \quad ,$$

$$\Phi_{k\,l} \in \{\mathcal{U}_L\}/\mathcal{D} \quad , \quad \hat{\Phi}_{\hat{k}\,\hat{l}} \in \{\mathcal{U}_R\}/\mathcal{D} \quad .$$

Here $\Phi_{k\,l}$ together $\hat{\Phi}_{\hat{k}\,\hat{l}}$ correspond to $\{\mathcal{B}\}$ while $\hat{\Psi}_{k\,\hat{l}}$ together with $\Psi_{\hat{k}\,l}$ correspond to $\{\mathcal{F}\}$.

In $\mathcal{GR}(2,2)$ we can use the following conventions and identities:

$$
\begin{aligned}
\mathrm{L}^{\mathrm{I}}\mathrm{R}^{\mathrm{J}} &= -\,\delta^{\mathrm{IJ}}\mathrm{I} \,+\, \epsilon^{\mathrm{IJ}}\, f^{*} \quad , \\
\mathrm{R}^{\mathrm{I}}\mathrm{L}^{\mathrm{J}} &= -\,\delta^{\mathrm{IJ}}\mathrm{I} \,+\, \epsilon^{\mathrm{IJ}}\, \hat{f}^{*} \quad , \\
Tr[f^{*}] \,=\, Tr[\hat{f}^{*}] &= 0 \quad , \\
(f^{*})^{2} \,=\, (\hat{f}^{*})^{2} &= -\,\mathrm{I} \quad .
\end{aligned}
$$

In order to define component fields we expand the $\mathcal{GR}(2,2)$ fields in the following manner:

$$
\begin{aligned}
\Phi_{k\,l} &= \tfrac{1}{2}\delta_{k\,l}B \,+\, \tfrac{1}{2}(f^{*})_{k\,l}\,(\partial_{\tau})^{-1}G \quad , \\
\Psi_{\hat{k}\,l} &= -\,\tfrac{1}{2}\mathrm{R}^{\mathrm{I}}_{\hat{k}l}\psi^{\mathrm{I}} \quad , \\
\widehat{\Phi}_{\hat{k}\,\hat{l}} &= \tfrac{1}{2}\hat{\delta}_{\hat{k}\hat{l}}A \,+\, \tfrac{1}{2}(\hat{f}^{*})_{\hat{k}\hat{l}}(\partial_{\tau})^{-1}F \quad , \\
\widehat{\Psi}_{k\,\hat{l}} &= \tfrac{1}{2}\mathrm{L}^{\mathrm{I}}_{k\,\hat{l}}\hat{\psi}^{\mathrm{I}} \quad ,
\end{aligned}
$$

$$
\begin{aligned}
\delta_Q A = &\ i\,\epsilon^{\rm I}\hat{\psi}^{\rm I} \ +\ i\,\bar{\epsilon}^{\rm I}\psi^{\rm I} \quad ,\\
\delta_Q B = &\ i\,\epsilon^{\rm I}\psi^{\rm I} \ -\ i\bar{\epsilon}^{\rm I}\hat{\psi}^{\rm I} \quad ,\\
\delta_Q \psi^{\rm I} = &\ \epsilon^{\rm I}\partial_\tau\, B \ +\ \bar{\epsilon}^{\rm I}\partial_\tau\, A \\
&\ -\ \epsilon^{\rm J}\epsilon^{\rm JI} G \ +\ \bar{\epsilon}^{\rm J}\epsilon^{\rm JI}\, F \quad ,\\
\delta_Q \hat{\psi}^{\rm I} = &\ -\ \epsilon^{\rm I}\partial_\tau\, A \ +\ \bar{\epsilon}^{\rm I}\,\partial_\tau B \\
&\ +\ \epsilon^{\rm J}\epsilon^{\rm JI}\, F \ +\ \bar{\epsilon}^{\rm J}\epsilon^{\rm JI}\, G \quad ,\\
\delta_Q F = &\ i\epsilon^{\rm I}\epsilon^{\rm IJ}\partial_\tau\,\hat{\psi}^{\rm J} \ +\ i\bar{\epsilon}^{\rm I}\epsilon^{\rm IJ}\partial_\tau\,\psi^{\rm J} \quad ,\\
\delta_Q G = &\ -i\epsilon^{\rm I}\epsilon^{\rm IJ}\partial_\tau\,\psi^{\rm J} \ -\ i\bar{\epsilon}^{\rm I}\epsilon^{\rm IJ}\partial_\tau\,\hat{\psi}^{\rm J}
\end{aligned}
$$

$$
\begin{aligned}
\delta_Q A &= -i\,\epsilon^1\,\hat{\psi}^1 - i\,\epsilon^2\,\hat{\psi}^2 + i\,\bar{\epsilon}^1\,\psi^1 + i\,\bar{\epsilon}^2\,\psi^2 \quad , \\
\delta_Q B &= i\,\epsilon^1\,\psi^1 + i\,\epsilon^2\,\psi^2 + i\,\bar{\epsilon}^1\,\hat{\psi}^1 + i\,\bar{\epsilon}^2\,\hat{\psi}^2 \quad , \\
\delta_Q \psi^1 &= \epsilon^1 \partial_\tau\, B + \epsilon^2\, G + \bar{\epsilon}^1\,\partial_\tau\, A - \bar{\epsilon}^2\, F \quad , \\
\delta_Q \psi^2 &= \epsilon^2\,\partial_\tau\, B - \epsilon^1\, G + \bar{\epsilon}^2\,\partial_\tau\, A + \bar{\epsilon}^1\, F \quad , \\
\delta_Q \hat{\psi}^1 &= -\,\epsilon^1\,\partial_\tau\, A - \epsilon^2\, F + \bar{\epsilon}^1\,\partial_\tau B - \bar{\epsilon}^2\, G \quad , \\
\delta_Q \hat{\psi}^2 &= -\,\epsilon^2\,\partial_\tau\, A + \epsilon^1\, F + \bar{\epsilon}^2\,\partial_\tau B + \bar{\epsilon}^1\, G \quad , \\
\delta_Q F &= i\epsilon^1\,\partial_\tau\,\hat{\psi}^2 - i\epsilon^2\,\partial_\tau\,\hat{\psi}^1 \\
&\quad + i\bar{\epsilon}^1\,\partial_\tau\,\psi^2 - i\bar{\epsilon}^2\,\partial_\tau\,\psi^1 \quad , \\
\delta_Q G &= -i\epsilon^1\,\partial_\tau\,\psi^2 + i\epsilon^2\,\partial_\tau\,\psi^1 \\
&\quad + i\bar{\epsilon}^1\,\partial_\tau\,\hat{\psi}^2 - i\bar{\epsilon}^2\,\partial_\tau\,\hat{\psi}^1 \quad .
\end{aligned}
$$

Colorized Transformation Laws

$$\delta_Q A = -i\,\epsilon\,\hat{\psi}^1 - i\,\epsilon\,\hat{\psi}^2 + i\,\epsilon\,\psi^1 + i\,\epsilon\,\psi^2 \ ,$$

$$\delta_Q B = i\,\epsilon\,\psi^1 + i\,\epsilon\,\psi^2 + i\,\epsilon\,\hat{\psi}^1 + i\,\epsilon\,\hat{\psi}^2 \ ,$$

$$\delta_Q \psi^1 = \epsilon\partial_\tau B + \epsilon\,G + \epsilon\,\partial_\tau A - \epsilon\,F \ ,$$

$$\delta_Q \psi^2 = \epsilon\,\partial_\tau B - \epsilon\,G + \epsilon\,\partial_\tau A + \epsilon\,F \ ,$$

$$\delta_Q \hat{\psi}^1 = -\epsilon\,\partial_\tau A - \epsilon\,F + \epsilon\,\partial_\tau B - \epsilon\,G \ ,$$

$$\delta_Q \hat{\psi}^2 = -\epsilon\,\partial_\tau A + \epsilon\,F + \epsilon\,\partial_\tau B + \epsilon\,G \ ,$$

$$\delta_Q F = i\,\epsilon\,\partial_\tau\,\hat{\psi}^2 - i\,\epsilon\,\partial_\tau\,\hat{\psi}^1 + i\,\epsilon\,\partial_\tau\,\psi^2 - i\,\epsilon\,\partial_\tau\,\psi^1 \ ,$$

$$\delta_Q G = -i\,\epsilon\,\partial_\tau\,\psi^2 + i\,\epsilon\,\partial_\tau\,\psi^1 + i\,\epsilon\,\partial_\tau\,\hat{\psi}^2 - i\,\epsilon\,\partial_\tau\,\hat{\psi}^1 \ .$$

Physical bosonic fields A and B engineering dimensions $-1/2$

Physical fermionic fields ψ^1, ψ^1, $\hat{\psi}^3$ and $\hat{\psi}^4$ engineering dimensions 0.

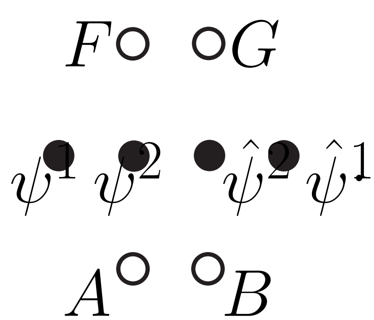
Auxiliary bosonic fields F and G engineering dimensions $+1/2$.

Height = Engineering Dimensions

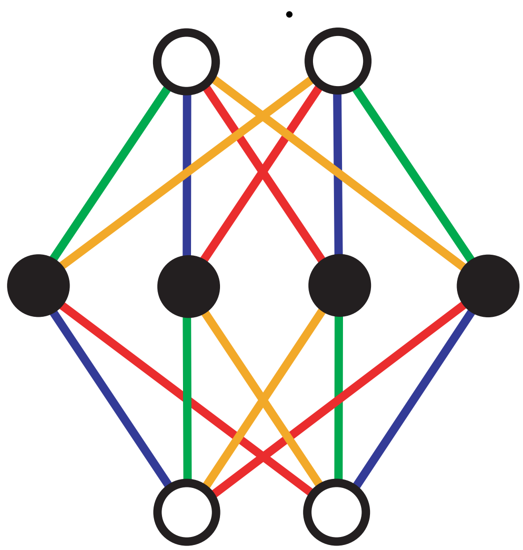
All nodes of the same height are drawn horizontally.

Nodes with different height assignments drawn accordingly.

'Skeleton' of the Adinkra



The next step is to introduce color edges which follow from colorized transformation laws

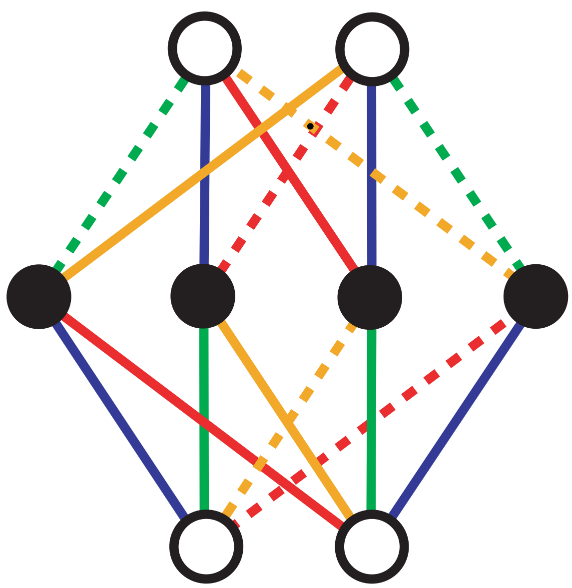


'Peacock mode' of the Adinkra

For some purposes, it is not necessary to display this level of detail. In this case, nodes may be 'collapsed' upon one another. For example, the fully collapsed version of this Adinkra is given by

,

where the numbers next to the nodes signify their respective multiplicities.



'Rampant peacock mode' of the Adinkra

Solid lines indicate positive coefficients

Dashed lines indicate negative coefficients.

Holomorphic Structure

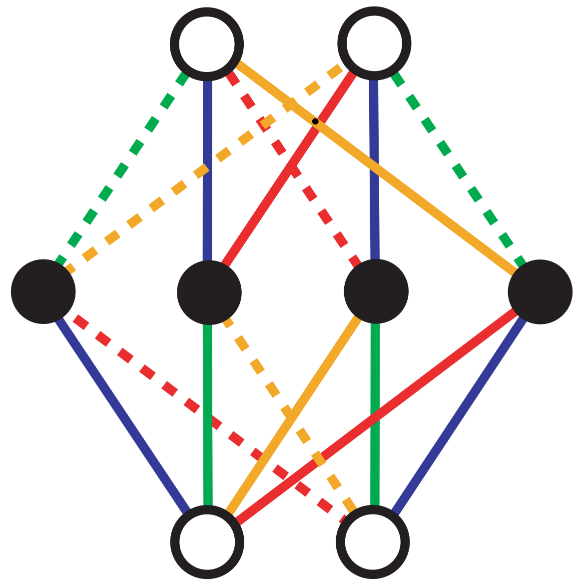
$$
\begin{aligned}
\delta_Q(A+iB) &= i(\epsilon+i\epsilon)(\psi^1+i\hat{\psi}^1) \quad , \\
&\quad +i(\epsilon+i\epsilon)(\psi^2+i\hat{\psi}^2) \quad , \\
\delta_Q(\psi^1+i\hat{\psi}^1) &= (\epsilon-i\epsilon)\partial_\tau(A+iB) \quad , \\
&\quad -(\epsilon+i\epsilon)(F+iG) \quad , \\
\delta_Q(\psi^2+i\hat{\psi}^2) &= (\epsilon-i\epsilon)\partial_\tau(A+iB) \quad , \\
&\quad +(\epsilon+i\epsilon)(F+iG) \quad , \\
\delta_Q(F+iG) &= -i(\epsilon-i\epsilon)\partial_\tau(\psi^1+i\hat{\psi}^1) \quad , \\
&\quad +i(\epsilon-i\epsilon)\partial_\tau(\psi^2+i\hat{\psi}^2) \quad .
\end{aligned}
$$

Taking the complex conjugate of these equations, we obtain

$$
\begin{aligned}
\delta_Q(A - iB) \; &= \; -i(\epsilon - i\epsilon)(\psi^1 - i\hat{\psi}^1) \quad , \\
& \quad\; -i(\epsilon - i\epsilon)(\psi^2 - i\hat{\psi}^2) \quad , \\
\delta_Q(\psi^1 - i\hat{\psi}^1) \; &= \; (\epsilon + i\epsilon)\partial_\tau(A - iB) \quad , \\
& \quad\; -(\epsilon - i\epsilon)(F - iG) \quad , \\
\delta_Q(\psi^2 - i\hat{\psi}^2) \; &= \; (\epsilon + i\epsilon)\partial_\tau(A - iB) \quad , \\
& \quad\; +(\epsilon - i\epsilon)(F - iG) \quad , \\
\delta_Q(F - iG) \; &= \; i(\epsilon + i\epsilon)\partial_\tau(\psi^1 - i\hat{\psi}^1) \quad , \\
& \quad\; -i(\epsilon + i\epsilon)\partial_\tau(\psi^2 - i\hat{\psi}^2) \qquad (1)
\end{aligned}
$$

which, by definition, corresponds to the 4D, $\mathcal{N} = 1$ anti-chiral multiplet.

Complex conjugation of the analytic form of the transformation laws corresponds



to the exchange of solid-red/solid-orange edges for dashed-red/dashed-orange edges and vice-versa.

One can start by considering any standard formulation of the 4D, $\mathcal{N} = 1$ chiral multiplet.

Followed by performing a toroidal reduction on a 0-brane and the results that emerge are isomorphic to those above.

1D model as the 'shadow' of the 4D, $\mathcal{N} = 1$ chiral multiplet.

The way in which the holomorphic parameters $((\epsilon + i\epsilon)$ and $(\epsilon + i\epsilon))$ and their anti-holomorphic conjugate parameters $((\epsilon - i\epsilon)$ and $(\epsilon - i\epsilon))$ appear in the transformation laws of is *exactly* as required for the fields to describe the dimensional reduction of the usual 4D, $\mathcal{N} = 1$ chiral multiplet on a 0-brane.

A Road to Higher Dimensions: "Root Superfields" in $\mathcal{GR}$(d, $\mathcal{N}$)

Any superfield expanded over matrices associated with $\mathcal{GR}(\mathrm{d},\mathcal{N})$ or $\mathcal{EGR}(\mathrm{d},\mathcal{N})$ has the forms

$$
\begin{aligned}
\Phi = &\ \phi(\tau)\,\mathbf{I} \ + \ \phi_{\,\mathrm{I}_1\,\mathrm{I}_2}(\tau)\, f^{\,\mathrm{I}_1\,\mathrm{I}_2} \\
&\ + \ \phi_{\,\mathrm{I}_1\,\mathrm{I}_2\,\mathrm{I}_3\,\mathrm{I}_4}(\tau)\, f^{\,\mathrm{I}_1\,\mathrm{I}_2\,\mathrm{I}_3\,\mathrm{I}_4} \ + \\
&\ \dots \quad , \\
\Psi = &\ \psi_{\,\mathrm{I}_1}(\tau)\,\hat{f}^{\,\mathrm{I}_1} \ + \ \psi_{\,\mathrm{I}_1\,\mathrm{I}_2\,\mathrm{I}_3}(\tau)\,\hat{f}^{\,\mathrm{I}_1\,\mathrm{I}_2\,\mathrm{I}_3} \\
&\ + \ \dots \quad .
\end{aligned}
$$

Definition of Root Superfield

$$
\begin{aligned}
\widetilde{\Phi} = &\ [(\partial_\tau)^{a_0}\phi]\,\mathbf{I} \ + \ [(\partial_\tau)^{a_2}\phi_{\,\mathrm{I}_1\,\mathrm{I}_2}]\, f^{\,\mathrm{I}_1\,\mathrm{I}_2} \\
&\ + \ [(\partial_\tau)^{a_4}\phi_{\,\mathrm{I}_1\,\mathrm{I}_2\,\mathrm{I}_3\,\mathrm{I}_4}]\, f^{\,\mathrm{I}_1\,\mathrm{I}_2\,\mathrm{I}_3\,\mathrm{I}_4} \ + \ \dots \quad , \\
\widetilde{\Psi} = &\ [(\partial_\tau)^{a_1}\psi_{\,\mathrm{I}_1}]\,\hat{f}^{\,\mathrm{I}_1} \\
&\ + \ [(\partial_\tau)^{a_3}\psi_{\,\mathrm{I}_1\,\mathrm{I}_2\,\mathrm{I}_3}]\,\hat{f}^{\,\mathrm{I}_1\,\mathrm{I}_2\,\mathrm{I}_3} \ + \ \dots \quad .
\end{aligned}
$$

for non-positive integers a_0, a_1, etc.

A Proposal for Holographic Encoding of Higher D

Consider the 2D, $\mathcal{N} = 1$ scalar superfield

$$\mathbf{X}(\theta^+, \theta^-, \tau, \sigma) = X(\tau, \sigma) + \theta^+ \psi_+(\tau, \sigma) + \theta^- \psi_-(\tau, \sigma) + i\,\theta^+ \theta^- F(\tau, \sigma) \quad .$$

SUSY Variation

$$\delta_Q X = \epsilon^+ \psi_+ + \epsilon^- \psi_- \quad ,$$
$$\delta_Q \psi_+ = i\epsilon^+ \partial_{\ddagger} X + i\epsilon^- F \quad ,$$
$$\delta_Q \psi_- = i\epsilon^- \partial_{=} X - i\epsilon^+ F \quad ,$$
$$\delta_Q F = - [\, \epsilon^+ \partial_{\ddagger} \psi_- - \epsilon^- \partial_{=} \psi_+ \,] \quad .$$

$$[\, \delta_{Q_1} ,\, \delta_{Q_2}] = -i2 [\, \epsilon_1^+ \epsilon_2^+ \partial_{\ddagger} + \epsilon_1^- \epsilon_2^- \partial_{=} \,] \quad .$$

Reduction to 1D yields

$$\mathbf{X}(\theta^+, \theta^-, \tau) = X(\tau) + \theta^+ \psi_+(\tau) + \theta^- \psi_-(\tau) + i\,\theta^+ \theta^- F(\tau) \quad .$$

Usual SUSY Variation (reduced)

$$\delta_Q X = \epsilon^+ \psi_+ + \epsilon^- \psi_- \quad ,$$

$$\delta_Q \psi_+ = i\epsilon^+ \partial_\tau X + i\epsilon^- F \quad ,$$

$$\delta_Q \psi_- = i\epsilon^- \partial_\tau X - i\epsilon^+ F \quad ,$$

$$\delta_Q F = - [\epsilon^+ \partial_\tau \psi_- - \epsilon^- \partial_\tau \psi_+] \quad .$$

$$[\delta_{Q_1} , \delta_{Q_2}] = -i2 [\epsilon_1^+ \epsilon_2^+ \partial_\tau + \epsilon_1^- \epsilon_2^- \partial_\tau] \quad .$$

Start with a root superfield valued over the normal part of $\mathcal{U}_L(2, 2) \oplus \mathcal{M}_L(2, 2)$

$$\widetilde{\Phi} = [(\partial_\tau)^{a_0} \phi] \mathbf{I} + [(\partial_\tau)^{a_2} \phi_{\mathrm{I}_1 \mathrm{I}_2}] f^{\mathrm{I}_1 \mathrm{I}_2} \quad ,$$

$$\widetilde{\Psi} = [(\partial_\tau)^{a_1} \psi_{\mathrm{I}_1}] \hat{f}^{\mathrm{I}_1} \quad .$$

Make identifications $a_0 = a_1 = 0, \; a_2 = -1$ and

$$\alpha^{\mathrm{I}} = (\epsilon^+, \epsilon^-) \quad , \quad \psi^{\mathrm{I}} = (\psi_+, \psi_-) \quad ,$$

$$\phi = X \quad , \quad \phi_{\mathrm{I}_1 \mathrm{I}_2} = \epsilon_{\mathrm{I}_1 \mathrm{I}_2} F \quad .$$

Same results obtained from algebraic 1D approach!

Conjecture I.

All superfields that provide a linear representation of spacetime supersymmetry in all dimensions can be represented as Clifford-algebraic root superfields.

EoM & $\mathcal{GR}(\mathbf{d}, \mathcal{N})$

$F = 0. \longrightarrow$

$$\partial_{\ddagger} \psi_{-} \; = \; 0 \quad , \quad \partial_{=} \psi_{+} \; = \; 0 \quad .$$

$\longrightarrow$

$$\psi_{-} \; = \; \psi_{-}(\sigma^{=}) \quad , \quad \psi_{+} \; = \; \psi_{+}(\sigma^{\ddagger}) \quad .$$

$\longrightarrow$

$$\partial_{\ddagger} \partial_{=} X \; = \; 0 \quad ,$$

whose solution is given by

$$X \; = \; X_L(\sigma^{\ddagger}) \; + \; X_R(\sigma^{=}) \quad .$$

$$
\begin{aligned}
\delta_Q \, [\, X_L(\sigma^{++}) \,] &= \; \epsilon^+ \, \psi_+(\sigma^{++}) \quad , \\
\delta_Q \, [\, \psi_+(\sigma^{++}) \,] &= \; i\epsilon^+ \, \partial_{++} \, X_L(\sigma^{++}) \quad , \\
\delta_Q \, [\, X_R(\sigma^{=}) \,] &= \; \epsilon^- \, \psi_-(\sigma^{=}) \quad , \\
\delta_Q \, [\, \psi_-(\sigma^{=}) \,] &= \; i\epsilon^- \, \partial_{=} \, X_R(\sigma^{=}) \quad .
\end{aligned}
$$

"What $\mathcal{GR}$(d, $\mathcal{N}$) Clifford-algebraic superfields can re-produce these results?"

X_L and ψ_+ form one representation and X_R and ψ_- form another.

Clifford-algebraic superfields that produces these transformation laws $\in \mathcal{EGR}(1,\, 1)$.

Conjecture II.

If an on-shell supermultiplet is embedded into a representation of $\mathcal{EGR}(\mathrm{d}_{\mathcal{N}},\, \mathcal{N})$, *then an off-shell representation of this supermultiplet is embedded into* $\mathcal{EGR}(\mathrm{d}_{2\mathcal{N}},\, 2\mathcal{N})$.

$\mathcal{GR}(\mathrm{d},\mathcal{N})$ in On-Shell SUSY Abelian VM

3D N-extended SUSY Abelian Theory

$$S_{3DVM} = \int d^3x \, \Big[-\tfrac{1}{4} F^{\underline{a}\,\underline{b}} F_{\underline{a}\,\underline{b}} + i\mathrm{d}^{-1} \lambda^{\alpha}{}_{k}{}^{\hat{k}} (\gamma^{\underline{a}})_{\alpha\beta} \partial_{\underline{a}} \lambda^{\beta}{}_{\hat{k}}{}^{k} + \mathrm{d}^{-1} (\partial^{\underline{a}} \mathcal{B}_{j}{}^{i})(\partial_{\underline{a}} \mathcal{B}_{i}{}^{j}) \Big]$$

Variations

$$\delta_Q \mathcal{B}_i{}^j = \epsilon^{\alpha \mathrm{I}} (\mathrm{L}_{\mathrm{I}})_k{}^{\hat{k}} \Big[\delta_i{}^k \lambda_{\alpha \hat{k}}{}^j - \mathrm{d}^{-1} \delta_i{}^j \lambda_{\alpha \hat{k}}{}^k \Big] \quad ,$$

$$\delta_Q \lambda_{\alpha \hat{k}}{}^k = i\epsilon^{\beta \mathrm{I}} (\mathrm{R}_{\mathrm{I}})_{\hat{k}}{}^j (\gamma^{\underline{d}})_{\alpha\beta} \Big[\partial_{\underline{d}} \mathcal{B}_j{}^k + \tfrac{1}{4} \mathrm{d}^{-1} \delta_j{}^k \epsilon_{\underline{d}}{}^{\underline{bc}} F_{\underline{bc}} \Big] \quad ,$$

$$\delta_Q A_{\underline{a}} = i\, \epsilon^{\alpha \mathrm{I}} (\mathrm{L}_{\mathrm{I}})_k{}^{\hat{k}} (\gamma_{\underline{a}})_{\alpha\beta} \lambda^{\beta}{}_{\hat{k}}{}^k \quad .$$

$$
\begin{aligned}
\mathcal{B}_i{}^j(x) = \; & \tfrac{1}{2}(f^{\,\mathrm{I_1 I_2}})_i{}^j \, \varphi_{\,\mathrm{I_1 I_2}}(x) \\
& + \; \tfrac{1}{4!}(f^{\,\mathrm{I_1 I_2 I_3 I_4}})_i{}^j \, \varphi_{\,\mathrm{I_1 I_2 I_3 I_4}}(x) \; + \; \cdots \\
& + \; \tfrac{1}{(2\mathrm{p})!}(f^{\,\mathrm{I_1 \cdots I_{2p}}})_i{}^j \, \varphi_{\,\mathrm{I_1 \cdots I_{2p}}}(x) \quad ,
\end{aligned}
$$

$$
\begin{aligned}
\lambda_{\alpha\,\hat{k}}{}^k(x) = \; & (\mathrm{R}^{\,\mathrm{I_1}})_{\hat{k}}{}^k \, \lambda_{\,\alpha\,\mathrm{I_1}}(x) \\
& + \; \tfrac{1}{3!}(f^{\,\mathrm{I_1 I_2 I_3}})_{\hat{k}}{}^k \, \lambda_{\,\alpha\,\mathrm{I_1 I_2 I_3}}(x) \; + \; \cdots \\
& + \; \tfrac{1}{(2\mathrm{q}+1)!}(f^{\,\mathrm{I_1 \cdots I_{2q+1}}})_{\hat{k}}{}^k \, \lambda_{\,\alpha\,\mathrm{I_1 \cdots I_{2q+1}}}(x) \quad .
\end{aligned}
$$

$$(A_a(x), \, \mathcal{B}_i{}^j(x) \,) \; \in \mathcal{U}_L$$

$$\lambda_{\alpha\,\hat{k}}{}^k(x) \; \in \mathcal{M}_L$$

Superspace gauge covariant derivatives

$$
\begin{aligned}
\nabla_{\underline{A}} &\equiv \; (\, \nabla_{\alpha\,\mathrm{I}}, \; \nabla_{\underline{a}} \,) \quad , \\
\nabla_{\alpha\,\mathrm{I}} &\equiv \; D_{\alpha\,\mathrm{I}} \; + \; i \, g \, \Gamma_{\alpha\,\mathrm{I}}{}^{\hat{\alpha}} \, t_{\hat{\alpha}} \quad , \\
\nabla_{\underline{a}} &\equiv \; \partial_{\underline{a}} \; + \; i \, g \, \Gamma_{\underline{a}}{}^{\hat{\alpha}} \, t_{\hat{\alpha}}
\end{aligned}
$$

$t_{\hat{\alpha}}$ denote a set of Abelian generators and satisfy $(t_{\hat{\alpha}})^* = - \, (t_{\hat{\alpha}})$.

$$
\begin{aligned}
[\,\nabla_{\alpha\,\mathrm{I}}\,,\ \nabla_{\beta\,\mathrm{J}}\,\} =&\ i\,2\,\delta_{\,\mathrm{I\,J}}\,(\gamma^{\underline{c}})_{\alpha\beta}\nabla_{\underline{c}} \\
&+\ i\,4\,g\,C_{\alpha\,\beta}\,(f_{\,\mathrm{I\,J}})_j{}^i\,\mathcal{B}_i{}^{j\,\hat{\alpha}}\,t_{\hat{\alpha}} \quad , \\
[\,\nabla_{\alpha\,\mathrm{I}}\,,\ \nabla_{\underline{b}}\,\} =&\ 2\,g\,(\gamma_{\underline{b}})_\alpha{}^\beta(\mathrm{L}_{\,\mathrm{I}})_k{}^{\hat{k}}\,\lambda_{\beta\,\hat{k}}{}^{k\,\hat{\alpha}}\,t_{\hat{\alpha}} \quad , \\
[\,\nabla_{\underline{a}}\,,\ \nabla_{\underline{b}}\,\} =&\ i\,g\,F_{\underline{a}\,\underline{b}}{}^{\hat{\alpha}}\,t_{\hat{\alpha}} \quad .
\end{aligned}
$$

Superspace Bianchi identifies satisfied if

$$
\begin{aligned}
(f_{\,\mathrm{I\,J}})_p{}^r\,(\mathrm{L}_{\,\mathrm{K}})_r{}^{\hat{q}} \ =&\ \ -\ \delta_{\,\mathrm{J\,K}}\,(\mathrm{L}_{\,\mathrm{I}})_p{}^{\hat{q}} \\
&\ +\ \delta_{\,\mathrm{I\,K}}\,(\mathrm{L}_{\,\mathrm{J}})_p{}^{\hat{q}} \\
&\ +\ (f_{\,\mathrm{I\,J\,K}})_p{}^{\hat{q}} \quad ,
\end{aligned}
$$

$$
\begin{aligned}
(f_{\,\mathrm{I\,J}})_i{}^k \ =&\ \tfrac{1}{2}\,[\ (\mathrm{L}_{\,\mathrm{I}})_i{}^{\hat{q}}\,(\mathrm{R}_{\,\mathrm{J}})_{\hat{q}}{}^k \\
&\ -\ (\mathrm{L}_{\,\mathrm{J}})_i{}^{\hat{q}}\,(\mathrm{R}_{\,\mathrm{I}})_{\hat{q}}{}^k\ ]
\end{aligned}
$$

Conjecture III.

The constraints to which all irreducible superfields in all dimensions are subjected insure that irreducible supermultiplets are also irreducible representations of the $\mathcal{GR}(\mathrm{d}, \mathcal{N})$ *algebra.*

Conjecture IV.

All superfields that provide an off-shell <u>linear</u> representation of $\mathcal{N}$*-extended spacetime supersymmetric field theory for D-dimensional Minkowski spaces (with* $D > 1$*) can be embedded in the representations of the* $C(\mathcal{N} + D - 1,\, 2)$ *Clifford-algebra with a projection of the dependence on the second temporal coordinate taken to zero.*

The 4D Chiral Multiplet & Beyond

$$\hat{\Psi}_{k\,\hat{l}} \in \{\mathcal{M}_L\} \quad , \quad \Psi_{\hat{k}\,l} \in \{\mathcal{M}_R\} \quad ,$$
$$\Phi_{k\,l} \in \{\mathcal{U}_L\} \quad , \quad \Phi_{\hat{k}\,\hat{l}} \in \{\mathcal{U}_R\} \quad .$$

Expand the $\mathcal{EGR}(2,2)$ fields in the following manner:

$$\begin{aligned}
\Phi_{k\,l} &= \tfrac{1}{2}\delta_{k\,l}B + \tfrac{1}{4}(\epsilon\cdot f)_{k\,l}\,(\partial_\tau)^{-1}G \quad , \\
\widehat{\Phi}_{\hat{k}\,\hat{l}} &= \tfrac{1}{2}\hat{\delta}_{\hat{k}\hat{l}}A + \tfrac{1}{4}(\epsilon\cdot\hat{f})_{\hat{k}\hat{l}}(\partial_\tau)^{-1}F \quad , \\
\Psi_{\hat{k}\,l} &= -\tfrac{1}{2}\mathrm{R}^{\mathrm{I}}_{\hat{k}l}\psi^{\mathrm{I}} \quad , \\
\widehat{\Psi}_{k\,\hat{l}} &= -\tfrac{1}{2}\mathrm{L}^{\mathrm{I}}_{k\,\hat{l}}\hat{\psi}^{\mathrm{I}} \quad .
\end{aligned}$$

Clifford-algebraic law produces the usual chiral multiplet transformations (after minor redefintions)!

Let us call the quartet of fields

$$\begin{matrix} & \Phi_{k_1\,k_2} & \\ \Psi_{\hat{k}_1\,k_2} & & \widehat{\Psi}_{k_1\hat{k}_2} \\ & \widehat{\Phi}_{\hat{k}_1\,\hat{k}_2} & \end{matrix}$$

a rank two representation. Obviously higher rank ones are possible.

The $n = 4$ representation seems appropriate to describe the 4D, $\mathcal{N} = 1$ VM

==============================

$\Phi_{k_1\, k_2\, k_3\, k_4}$

$\Psi_{\hat{k}_1\, k_2\, k_3\, k_4} \quad \Psi_{k_1\, \hat{k}_2\, k_3\, k_4} \quad \Psi_{k_1\, k_2\, \hat{k}_3\, k_4} \quad \Psi_{k_1\, k_2\, k_3\, \hat{k}_4}$

$\Phi_{\hat{k}_1\, \hat{k}_2\, k_3\, k_4} \quad \Phi_{\hat{k}_1\, k_2\, \hat{k}_3\, k_4} \quad \Phi_{\hat{k}_1\, k_2\, k_3\, \hat{k}_4}$

$\Phi_{k_1\, \hat{k}_2\, \hat{k}_3\, k_4} \quad \Phi_{k_1\, \hat{k}_2\, k_3\, \hat{k}_4} \quad \Phi_{k_1\, k_2\, \hat{k}_3\, \hat{k}_4}$

$\Psi_{\hat{k}_1\, \hat{k}_2\, \hat{k}_3\, k_4} \quad \Psi_{\hat{k}_1\, \hat{k}_2\, k_3\, \hat{k}_4} \quad \Psi_{\hat{k}_1\, k_2\, \hat{k}_3\, \hat{k}_4} \quad \Psi_{k_1\, \hat{k}_2\, \hat{k}_3\, \hat{k}_4}$

$\Phi_{\hat{k}_1\, \hat{k}_2\, \hat{k}_3\, \hat{k}_4}$

It can be seen from our previous table the rank four tensor possesses "components" tensors for the representation fall into the pattern of the binomial coefficients. This is a feature that is expected to occur in general.

We conjecture that all 4D, $\mathcal{N} = 1$ off-shell supermultiplets are associated with higher even-rank tensors of $\mathcal{EGR}(2,2)$.

$$s_{max} \;=\; \tfrac{1}{2}\,Rank(tensor) \;=\; \tfrac{1}{2}\,n$$

Understanding in detail how higher $\mathcal{N}$-extended and higher D supermultiplets are associated with these Clifford algebras is a future task.

We end the talk by showing that there is already evidence that on-shell 4D, $\mathcal{N} = 8$ SG shows such a relation.

$\mathcal{GR}(8,\,8)$ enveloping algebras

$$\mathcal{U}_L = \{\, \mathbf{I}, f_{\rm I\,J}, f^-_{\rm I_1 I_2 I_3 I_4}\}\, , \mathcal{U}_R = \{\mathbf{I}, \widehat{f}_{\rm I\,J}, \widehat{f}^+_{\rm I_1 I_2 I_3 I_4}\}\ ,$$
$$\mathcal{M}_L = \{\ f_{\rm I}, \ f_{\rm I\,J\,K}\ \}\ ,\ \mathcal{M}_R\ = \{\ \widehat{f}_{\rm I}\ ,\ \widehat{f}_{\rm I\,J\,K}\ \}\quad .$$

Algebraic element	Spin	Degeneracy
$\mathcal{U}_L(\mathbf{I})$	2	1
$\mathcal{U}_R(\mathbf{I})$	2	1
$\mathcal{M}_L(f_{\rm I})$	3/2	8
$\mathcal{M}_R(\widehat{f}_{\rm I})$	3/2	8
$\mathcal{U}_L(f_{\rm IJ})$	1	28
$\mathcal{U}_R(\widehat{f}_{\rm IJ})$	1	28
$\mathcal{M}(f_{\rm I_1 I_2 I_3})$	1/2	56
$\mathcal{M}_R(\widehat{f}_{\rm I_1 I_2 I_3})$	1/2	56
$\mathcal{U}_L(f^-_{\rm I_1 I_2 I_3 I_4})$	0	35
$\mathcal{U}_R(\widehat{f}^+_{\rm I_1 I_2 I_3 I_4})$	0	35

Table III

$$s \equiv (2 - \tfrac{1}{2}p) \quad ,$$

where p is the rank of the generator.

Adinkras can be obtain for *all* supersymmetrical representations by applying a projection that retains only the temporal dependence of the higher dimensional field.

In this way Adinkras *and* the $\mathcal{GR}(\mathrm{d}_{\mathcal{N}}, \mathcal{N})$ algebras are universal to supersymmetrical theories. In fact the $\mathcal{GR}(\mathrm{d}_{\mathcal{N}}, \mathcal{N})$ algebras appear to play a role for supersymmetrical theories that is analogous to that of the Weyl 'little group' for non-supersymmetrical field theories.

The Appearance of Topology of Representations

It was likely not obvious, but the one explicitly studied representation strongly suggests that there is an important role to be played by topology.

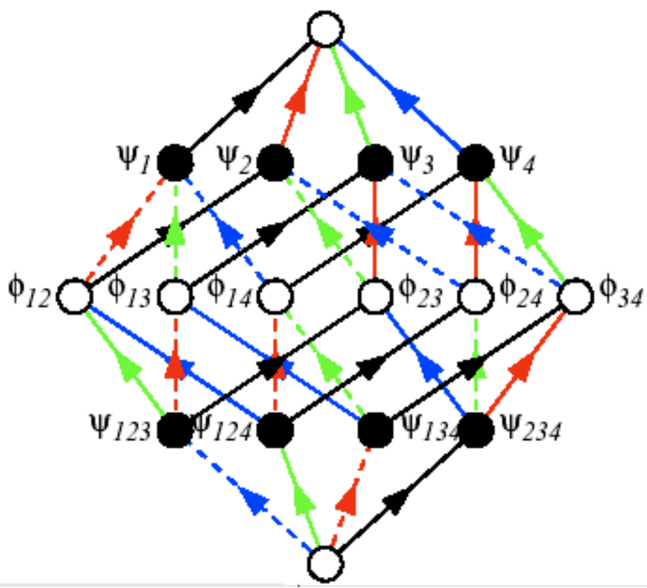
Closed loops can be constructed on the demonstrated Adinkras.

A particularly interesting class of loops are those which describe closed paths that only utilize two colors. For every such path one encounters an odd number of minus signs. Thus, the product of signs seems to be similar to a topological index.

To date our DFGHIL collaboration has found that this rule is violated in at least one case...the so-called 'hypermultiplet.' The violation of the topological rule seem correlated with the appearance of a structure known as ‘an off-shell central charge.’

Presently the collaboration is studying the manner in which the topology of Adinkras is correlated with the appearance of new representations.

For example, the only other topology that is encountered in the case of $\mathcal{N} = 4$ has the structure



,

An evaluation of this in terms of its component field structure shows that it is precisely the shadow of the 4D $\mathcal{N} = 1$ vector multiplet. These two distinct topologies stand in correspondence to the decomposition the of scalar superfields of 4D $\mathcal{N} = 1$ theories.

It seems very likely that the problem of finding all irreducible off-shell and linear representation is equivalent to the study of the topologies of a certain class of Adinkras. I hope to be able to report on this in the future.

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