Truncations of random unitary matrices, Selberg-type integrals and colour-flavour transformations

[Short title: An exercise in Schur function expansion]

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Schur functions

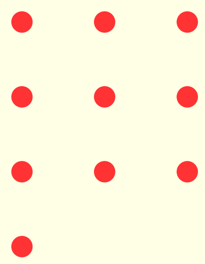
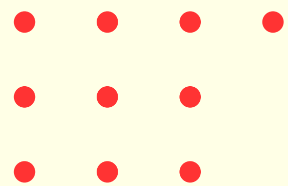
homogeneous symmetric polynomials indexed by partitions $\lambda = (\lambda_1, \ldots \lambda_m)$

$$s_\lambda(x_1, \ldots, x_m) = \det(x_i^{m+\lambda_j-j}) / \det(x_i^{m-j})$$

Two examples:

- If $\lambda = (r)$ then s_λ is the complete symmetric function of degree r.
- If $\lambda = (1, \ldots, 1)$ then s_λ is the elementary symmetric function of degree r.

Conjugate partition λ', defined in terms of Ferrer's diagram.

The conjugate partition of  is 

If $\lambda = (r)$ then $\lambda' = (1, \ldots, 1)$ (and vice versa).

Examples of Schur function expansions

By convention, if M is a matrix with eigvs x_j then $s_\lambda(M) = s_\lambda(x_1, \ldots, x_m)$.

- Cauchy identity

$$\det(I - L \otimes M)^{-1} = \sum_\lambda s_\lambda(L) s_\lambda(M)$$

- Dual Cauchy identity

$$\det(I + L \otimes M) = \sum_\lambda s_\lambda(L) s_{\lambda'}(M)$$

- Exponential function (e.g. Balantekin ’00, Orlov ’04)

$$e^{\mathrm{Tr}\, M} = \sum_\lambda c_\lambda s_\lambda, \qquad c_\lambda = s_\lambda(1_m) \prod_{j=1}^{m} \frac{(m-j)!}{(m+\lambda_j-j)!}, \quad m \geq \dim M.$$

Why expanding in Schur functions?

s_λ are irreducible characters and, as a consequence, are orthogonal

$$\int_{U(m)} s_\lambda(U)\overline{s_\mu(U)}d\mu_H(U) = \delta_{\lambda,\mu}, \qquad \text{integration measure - Haar}$$

i.e. c_λ are “Fourier”-coefficients (all identities on Slide 3 can be derived in this way).

More generally,

$$\int_{U(m)} s_\lambda(AU)\overline{s_\mu(BU)}d\mu_H(U) = \delta_{\lambda,\mu}\frac{s_\lambda(AB^*)}{s_\lambda(I_m)} \tag{1}$$

and also

$$\int_{U(m)} s_\lambda(AUBU^*)d\mu_H(U) = \frac{s_\lambda(A)s_\lambda(B)}{s_\lambda(I_m)}. \tag{2}$$

Matrix distributions

Consider rectangular matrices $Q \in \mathbb{C}^{n\times m}$ and square matrices $Z \in \mathbb{C}^{m\times m}$.

Assume $n \geq m$.

- Jacobi type (B) (a.k.a. Beta type I in multivariate stats)

$$\boxed{d\mu^B_{N,n\times m}(Q) \propto \det(I_m - Q^*Q)^{N-n-m}(dQ), \qquad Q^*Q \leq I_m,}$$

N is real and $N \geq n+m$. Note that $\langle s_\lambda(Q^*Q)\rangle_B$ reduces to

$$\int_{Z^*Z\leq I_m} s_\lambda(Z^*Z)\det(Z^*Z)^{n-m}\det(I_m - Z^*Z)^{N-n-m}(dZ)$$

- Jacobi type (F) (a.k.a. Beta type II in multivariate stats)

$$\boxed{d\mu^F_{N,n\times m}(Q) \propto \det(I_m + Q^*Q)^{-N-n-m}(dQ), \qquad Q \in \mathbb{C}^{n\times m},}$$

N is real and $N \geq 0$.

Jacobi type (B) describes distribution of truncations of CUE matrices

Let $d\rho_{N,n\times m}$ be the projection of the normalised Haar measure on $U(N)$ onto the matrix ball $Q^*Q \leq I_m$ under the map $U \mapsto Q$ where Q is the top left $n \times m$ block of U.

If $N \geq n+m$ then (Fyodorov's talk)

$$\boxed{d\rho_{N,n\times m}(Q) \propto \det(I_m - Q^*Q)^{N-n-m}(dQ), \qquad Q^*Q \leq I_m}$$

What if $N < n+m$? Then $d\rho_{N,n\times m}(Q)$ is supported on the boundary of $Q^*Q \leq I_m$. Have integration formula: for *invariant* f,

$$\int_{Q^*Q\leq I_m} f(Q^*Q)d\rho_{N,n\times m}(Q) = \text{const.}\times$$

$$\int_{Z^*Z\leq I_{N-n}} f\begin{pmatrix} Z^*Z & 0 \\ 0 & I \end{pmatrix} \det(I_{N-n} - Z^*Z)^{n+m-N} \det(Z^*Z)^{n-m}\,(dZ)$$

Two useful identities $A \in \mathbb{C}^{n\times n}, \quad B \in \mathbb{C}^{m\times m}$

- Jacobi type (B). For integer $N \geq n+m$

$$\int_{Q^*Q\leq I_m} s_\lambda(AQBQ^*)d\mu^B_{N,n\times m}(Q) = \frac{s_\lambda(A)s_\lambda(B)}{s_\lambda(I_N)} \qquad \text{(UIB)}$$

and, more generally, for integer $N \geq n, m$

$$\int s_\lambda(AQBQ^*)d\rho_{N,n\times m}(Q) = \frac{s_\lambda(A)s_\lambda(B)}{s_\lambda(I_N)} \qquad \text{(UIB')}$$

- Jacobi type (F). For integer $N \geq 0$

$$\int_{\mathbb{C}^{n\times m}} s_\lambda(AQBQ^*)d\mu^F_{N,n\times m}(Q) = \frac{s_\lambda(A)s_\lambda(B)}{s_{\lambda'}(I_N)} \qquad \text{(UIF)}$$

(UIB') is a simple corollary of the invariance of $d\rho_{N,n\times m}$ wrt right and left multiplication by unitary matrices and integration formula (2).

Selberg integrals

(UIB) can be obtained from the Selberg type integral ($J_\lambda^{1/\gamma}$ are Jack polynomials)

$$\int_0^1 \cdots \int_0^1 J_\lambda^{\frac{1}{\gamma}}(x_1,\ldots,x_m) \prod_{j=1}^{m} x_j^{p-1}(1-x_j)^{q-1} \prod_{1\le i<j\le m} |x_i - x_j|^{2\gamma} \prod_{j=1}^{m} dx_j$$

evaluated by Kadell '88 (for $J_\lambda^1 = s_\lambda$), Kadell '97 (general), Yan '92, Kaneko '93.

$< J_\lambda^{\frac{1}{\gamma}} >$ for Jacobi type (F) weight yet to be evaluated.

$\gamma = 1$: (Schur functions) the integral in both cases, (B) and (F), can be evaluated by reducing it to binomial determinants (Fyodorov & K '06). (UIF) can be obtained from Selberg type (F).

$\gamma = 1/2$ (zonal polynomials): the type (B) integral evaluated by Constantine '63 and type (F) can be evaluated following this calculation.

Berezin's reproducing kernels

Matrices are $n \times m$, complex.

- Schur fnc expansion for $\det(I-M)^{-a}$, $a>0$, and Selberg type (B) yield

$$\int_{Q^*Q<I_m} \frac{d\mu^B_{N,n\times m}(Q)}{\det(I_n-PQ^*)^N \det(I_n-QR^*)^N} = \frac{1}{\det(I_n-PR^*)^N}.$$

  Holds for real $N \geq n+m$. (Berezin '75)

- Schur fnc expansion for $\det(I+M)^N$, integer N, and Selberg type (F) yield

$$\int_{\mathbb{C}^{n\times m}} \det(I_m+PQ^*)^N \det(I_m+QR^*)^N \, d\mu^F_{N,n\times m}(Q) = \det(I_m+PR^*)^N.$$

  Holds for integer $N \geq 0$. (Berezin '75 by a different method).

Angular integrals

N, m, n are positive integers (Think of $N \gg n, m$)

Matrices A, B, U are $N \times N$. Matrices Q are $n \times m$.

By Cauchy identities,

- (UIB) is equivalent to

$$\int_{U(N)} \frac{d\mu_H(U)}{\det(I_N - AU)^m \det(I_N - U^* B^*)^n} = \int_{Q^*Q \leq I_m} \frac{d\rho_{N,n\times m}(Q)}{\det(I - Q^*Q \otimes B^*A)},$$

$$1 \leq m \leq n \leq N.$$

- (UIF) is equivalent to

$$\int_{U(N)} \det(I_N + AU)^m \det(I_N + U^* B^*)^n d\mu_H(U) =$$

$$\int_{\mathbb{C}^{n\times m}} \det(I + Q^*Q \otimes B^*A) d\mu^F_{N,n\times m}(Q), \qquad 1 \leq m \leq n.$$

Application to spectral determinants of matrices with complex eigenvalues

Random matrices $W \in \mathbb{C}^{N\times N}$, with invariant ensemble distribution (e.g. $e^{-N\,\mathrm{Tr}\,V(W^*W)}(dW)$).

Note that joint pdf of eigenvalues is rarely known for such matrices.

In view of unitary invariance,

$$\langle |\det(I_N + zW)|^{2m}\rangle_W \propto$$

$$\int_{\mathbb{C}^{m\times m}} \langle \det(I + |z|^2 Q^* Q \otimes W^* W)\rangle_W \det(I_m + Q^* Q)^{-N-2m} (dQ)$$

Thus, integration over angular part of W can be traded for Jacobi average over $m \times m$ matrices Q. Advantage - now have Hermitian matrices W^*W, can apply orthogonal polynomial technique, etc.

Structure - Hankel determinants. Matrix elements are integrals involving orthogonal polynomials.

Colour-flavour transformations (Zirnbauer '96)

- Bosonic version. $\{\vec{x}_j\}_{j=1}^m$ and $\{\vec{y}_j\}_{j=1}^m$ are two sets of vectors in $\mathbb{C}^N$. If $2m \leq N$ then

$$\int_{U(N)} e^{\sum_{j=1}^m (\vec{y}_j^* U \vec{x}_j + \vec{x}_j^* U^* \vec{y}_j)} \, d\mu_H(U) =$$

$$\int_{Q^*Q \leq I_m} e^{\sum_{j,k=1}^m (Q_{jk} \vec{x}_k^* \vec{x}_j + (Q^*)_{jk} \vec{y}_k^* \vec{y}_j)} \, d\mu^B_{N,m\times m}(Q)$$

- Fermionic version. Now $\vec{\chi}_j$, $\vec{\psi}_j$, $\vec{\chi}_j^*$ and $\vec{\psi}_j^*$ are N-component vectors with anti-commuting components. For any m have

$$\int_{U(N)} e^{\sum_{j=1}^m (\vec{\chi}_j^* U \vec{\psi}_j + \vec{\psi}_j^* U^* \vec{\chi}_j)} \, d\mu_H(U) =$$

$$\int_{\mathbb{C}^{m\times m}} e^{\sum_{j,k=1}^m (Q_{jk} \vec{\chi}_k^* \vec{\chi}_j - (Q^*)_{jk} \vec{\psi}_k^* \vec{\psi}_j)} \, d\mu^F_{N,m\times m}(Q)$$

Zirnbauer: algebraic/geometric approach, other classical groups, SUSY variant. In the RMT context useful for evaluating CUE averages of determinantal products.

bCFT and truncations of CUE

X and Y are $N \times m$ with columns $\vec{x}_j$ and $\vec{y}_j$. XY^* has rank m (generically). Hence link to truncations of CUE.

By Schur fnc expansion and (UIB'), have

$$\int_{U(N)} e^{\mathrm{Tr}(XY^*U+U^*X^*Y)}\, d\mu_H(U) = \int_{Q^*Q \le I_m} e^{\mathrm{Tr}(QX^*X+Q^*Y^*Y)} d\rho_{N,m\times m}(Q).$$

This is another form of bCFT, but now in the extended range $m \le N$.

If $2m \le N$ then $d\rho_{N,m\times m} = d\mu^B_{N,m\times m}$ and we are back to Zirnbauer.

If $N < 2m < 2N$ then $d\rho_{N,m\times m}$ is supp by the set $\mathrm{rk}(I_m - Q^*Q) = N - m$ on the boundary of $Q^*Q \le I_m$ in $\mathbb{C}^{m\times m}$. This set can be parametrized by

$$Q_{UZV^*} = U \begin{pmatrix} Z & 0 \\ 0 & I_{2m-N} \end{pmatrix} V^*, \qquad Z^*Z < I_{N-m},\ U, V \in U(m)$$

bCFT in the extended range

Using matrices Q_{UZV^*}, have bCFT in the range $N < 2m < 2N$:

$$\int_{U(N)} e^{\mathrm{Tr}(Y^*UX+X^*U^*Y)} d\mu_H(U) =$$

$$\int_{U(m)} d\mu_H(U) \int_{U(m)} d\mu_H(V) \int_{Z^*Z \leq I_{N-m}} e^{\mathrm{Tr}(X^*XQ_{UZV^*}+Q^*_{UZV^*}Y^*Y)}\, d\mu^B_{N,N-m}(Z)$$

CFTs and Selberg type integrals (linked via Schur function expansions):

- bCFT implies (UIB') and vice versa
- fCFT implies (UIF). Is the converse true?

Regularised inverse determinants

$$R_\varepsilon(A^*A) = \int_{U(N)} \frac{d\mu_H(U)}{\det[(I_N + AU)(I_N + AU)^* + \varepsilon^2 I_N]^m}$$

Non-trivial even for $m = 1$. Direct application of bCFT runs into a problem (diverging integrals). Schur functions do not help.

A deformed version of CFT: integration over $Q^*Q \leq I_m$ can be replaced by integration over a pair of Hermitian matrices $Q_1 = TPT^*, Q_2 = (T^*)^{-1}PT^{-1}$, where $T \in Gl_m(\mathbb{C})$ and $P = \mathrm{diag}(p_1, \ldots, p_m)$, $|p_j| \leq 1$. ”Volume element”

$$(dQ_1 dQ_2) = d\mu_H(T) \prod_{j<k} (p_j^2 - p_k^2)^2 \prod_j p_j dp_j$$

Expression for $R_\varepsilon(A^*A)$ in simplest case $m = 1$:

$$\frac{N-1}{2\pi i} \int_0^1 (1-t)^{N-2} dt \int_{-\infty}^{+\infty} \frac{dx}{x} \frac{1}{\det\left[AA^* + (\varepsilon^2 - t)\, I - i\varepsilon\sqrt{t}\left(x + \frac{1}{x}\right) I\right]}.$$

References:

1. Balantekin A B 2000 Character expansions, Itzykson-Zuber integrals, and the QCD partition function *Phys Rev* D(3) **62** 085017.
2. Berezin F A 1975 Quantization in complex symmetric spaces *Izv Akad Nauk SSSR, Ser Math* **39** 363–402; English translation: *Math USSR-Izv* **9 (1975)**, no. 2, 341–379 (1976)
3. Kadell K W J 1997 The Selberg-Jack symmetric functions *Adv Math* **130** 33 – 102.
4. Kaneko J 1993 Selberg integrals and hypergeometric functions associated with Jack polynomials. *SIAM J Math Anal* **24** 1086–1110.
5. Orlov A Yu 2004 New solvable matrix integrals. *Internat J Modern Phys* **A 19**, May, suppl., 276–293.
6. Yan Z M 1992 A class of generalized hypergeometric functions in several variables. *Canad J Math* **44** 1317–1338.
7. Zirnbauer M R 1996 Supersymmetry for systems with unitary disorder: Circular ensembles *J Phys A* **29** 7113 – 7136.