

# MODULE CATEGORIES OVER REPRESENTATIONS OF $SL_q(2)$ AND GRAPHS

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ABSTRACT. We classify module categories over the category of representations of quantum  $SL(2)$  in a case when  $q$  is not a root of unity. In a case when  $q$  is a root of unity we classify module categories over the semisimple subquotient of the same category.

## 1. INTRODUCTION

Let  $q$  be a root of unity of even order  $N > 4$ . Let  $\mathcal{C}_q$  be the corresponding fusion category of representations of the quantum group  $SL_q(2)$ . It is known (see [Oc],[KO] and references therein) that indecomposable semisimple module categories over  $\mathcal{C}_q$  correspond to the ADE Dynkin diagrams with Coxeter number  $h = N/2$ . This fact may be viewed as the “quantum McKay’s correspondence”. More specifically, the module categories in question may be viewed as “quantum finite subgroups in  $SL_q(2)$ ”, by analogy with finite subgroups of  $SL(2)$ , which define module categories over  $\text{Rep}(SL(2))$  and are parametrized by the ADE affine Dynkin diagrams by virtue of the classical McKay’s correspondence.

In this paper, we generalize this picture to the case of any nonzero complex number  $q$ , not equal to  $\pm i$ . Namely, let  $q$  be such a number. If  $q = \pm 1$  or  $q$  is not a root of unity, let  $\mathcal{C}_q$  denote the category of representations of the quantum group  $SL_q(2)$ . If  $q$  is a root of unity such that  $q^4 \neq 1$ , we let  $\mathcal{C}_q$  denote the fusion category attached to the quantum group  $SL_q(2)$ . We classify indecomposable semisimple module categories over  $\mathcal{C}_q$  with finitely many simple objects. It turns out that such module categories are parametrized by connected graphs equipped with bilinear forms satisfying some relations. In the case when  $q$  is a root of unity of even order  $N > 4$ , this easily yields the classification of [Oc],[KO]; so in particular we obtain a very simple proof of the result of [KO], which does not involve vertex algebras and conformal inclusions (in fact, this proof is close to the original approach of [Oc]).

A striking property of our classification is that while all connected graphs do appear, trees appear only for special values of  $q$ , namely such that  $-q - q^{-1}$  is an eigenvalue of the adjacency matrix that admits an eigenvector with nonvanishing entries. Thus we discover somewhat unexpected “combinatorial” peculiarities of  $SL_q(2)$  at algebraic special values of  $q$  which are not roots of unity.

On the contrary, we show that graphs with cycles appear for generic  $q$  (or, equivalently, over  $\overline{\mathbb{C}(q)}$ ). This explains why the only finite subgroups of  $SL(2)$  which admit a continuous quantum deformation (into subgroups of  $SL_q(2)$ ) are  $\mathbb{Z}/n\mathbb{Z}$ : for them, the corresponding affine Dynkin graph has a cycle (type  $A_{n-1}$ ), while for other cases (types  $\tilde{D}_n, \tilde{E}_n$ ), this graph is a tree.

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## 2. MAIN EQUATION

2.1. **Quantum  $SL(2)$ .** We will work over the field  $\mathbb{C}$  of complex numbers. Let  $q \in \mathbb{C}$  be a nonzero number,  $q^2 \neq -1$ . Recall (see e.g. [K]) that the Hopf algebra  $SL_q(2)$  is defined by generators  $a, b, c, d$  and relations:

$$\begin{aligned} ba &= qab, \quad db = qbd, \quad ca = qac, \quad dc = qcd, \quad bc = cb, \\ ad - da &= (q^{-1} - q)bc, \quad ad - q^{-1}bc = 1, \\ \Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \\ \varepsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}. \end{aligned}$$

Let  $\tilde{\mathcal{C}}_q$  denote the tensor category of finite dimensional comodules over  $SL_q(2)$ .

Let  $\mathbf{1} \in \tilde{\mathcal{C}}_q$  denote the unit object and let  $V \in \tilde{\mathcal{C}}_q$  be a two dimensional comodule  $V$  with the basis  $x, y$  and the coaction given by

$$\Delta_V \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix}.$$

The following well known property of the object  $V$  will be crucial for us:

The object  $V \in \tilde{\mathcal{C}}_q$  is selfdual, moreover for any isomorphism  $\phi : V \rightarrow V^*$  the composition

$$\mathbf{1} \xrightarrow{coev_V} V \otimes V^* \xrightarrow{\phi \otimes \phi^{-1}} V^* \otimes V \xrightarrow{ev_V} \mathbf{1} \quad (1)$$

equals to  $-(q + q^{-1})\text{id}_{\mathbf{1}}$ .

Indeed, let  $\delta_x, \delta_y \in V^*$  be the dual basis to  $x, y \in V$ . By the definition

$$\Delta_{V^*} \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix} = \begin{pmatrix} d & -q^{-1}c \\ -qb & a \end{pmatrix} \otimes \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix}.$$

It is easy to check that the map  $\phi(x) = \delta_y, \phi(y) = -q\delta_x$  is an isomorphism of comodules. Finally, the composition in (1) equals to

$$1 \mapsto x \otimes \delta_x + y \otimes \delta_y \mapsto \delta_y \otimes (-q^{-1}y) + (-q\delta_x) \otimes x \mapsto -q^{-1} - q.$$

Note that since  $V \in \tilde{\mathcal{C}}_q$  is irreducible, the isomorphism  $\phi$  is unique up to scaling, (so composition (1) equals  $-(q + q^{-1})\text{id}_{\mathbf{1}}$  for any  $\phi$ ). From now on we fix a choice of such isomorphism.

Equivalently, we can replace the isomorphism  $\phi$  by two maps

$$\alpha := (\text{id}_V \otimes \phi^{-1}) \circ coev_V : \mathbf{1} \rightarrow V \otimes V, \quad \beta := ev_V \circ (\phi \otimes \text{id}_V) : V \otimes V \rightarrow \mathbf{1}$$

such that:

1) The compositions  $V \xrightarrow{\alpha \otimes \text{id}_V} V \otimes V \otimes V \xrightarrow{\text{id}_V \otimes \beta} V$  and  $V \xrightarrow{\text{id}_V \otimes \alpha} V \otimes V \otimes V \xrightarrow{\beta \otimes \text{id}_V} V$  both equal to  $\text{id}_V : V \rightarrow V$ .

2) The composition  $\mathbf{1} \xrightarrow{\alpha} V \otimes V \xrightarrow{\beta} \mathbf{1}$  equals to  $-(q + q^{-1})\text{id}_{\mathbf{1}}$ .

Indeed, the map  $\phi$  can be reconstructed from the pair  $(\alpha, \beta)$  as the composition  $V \xrightarrow{\text{id}_V \otimes coev_V} V \otimes V \otimes V^* \xrightarrow{\beta \otimes \text{id}_{V^*}} V^*$ .

**2.2. Turaev's construction: the generic case.** Recall that in the case when  $q$  is not a root of unity or  $q = \pm 1$  the category  $\tilde{\mathcal{C}}_q$  is semisimple and we have a unique isomorphism of the Grothendieck rings (as based rings)  $\text{Gr}(\tilde{\mathcal{C}}_q) \simeq \text{Gr}(\tilde{\mathcal{C}}_1) = \text{Gr}(\text{Rep}(SL(2)))$ , see e.g. [K]. In other words, the category  $\tilde{\mathcal{C}}_q$  has exactly one simple comodule in each dimension and tensor products of simple comodules are decomposed in the same way as for  $SL(2)$ .

In Chapter XII of [T] V. Turaev gave a topological construction of the category  $\tilde{\mathcal{C}}_q$ . We reformulate his results in the following way:

**Theorem 2.1.** ([T]) *Assume that  $q$  is not a root of unity or  $q = \pm 1$ . The triple  $(\tilde{\mathcal{C}}_q, V, \phi)$  has the following universal property: let  $\mathcal{D}$  be an abelian monoidal category, let  $W \in \mathcal{D}$  be a right rigid object and  $\Phi : W \rightarrow W^*$  be an isomorphism such that the composition morphism*

$$\mathbf{1} \xrightarrow{\text{coev}_W} W \otimes W^* \xrightarrow{\Phi \otimes \Phi^{-1}} W^* \otimes W \xrightarrow{\text{ev}_W} \mathbf{1} \quad (2)$$

*equals to  $-(q+q^{-1})id_{\mathbf{1}}$ . Then there exists a unique tensor functor  $F : \tilde{\mathcal{C}}_q \rightarrow \mathcal{D}$  such that  $F(V) = W$  and  $F(\Phi) = \phi$ .*

**Sketch of proof.** We will freely use the notation from Chapter XII of [T]. Let  $\bar{\alpha} = (\text{id}_W \otimes \Phi^{-1}) \circ \text{coev}_W : \mathbf{1} \rightarrow W \otimes W$ ,  $\bar{\beta} := \text{ev}_W \circ (\Phi \otimes \text{id}_W) : W \otimes W \rightarrow \mathbf{1}$ . Obviously, the morphisms  $\bar{\alpha}$  and  $\bar{\beta}$  induce the homomorphisms  $E_{k,l} \rightarrow \text{Hom}(W^{\otimes k}, W^{\otimes l})$  compatible with the compositions (here  $E_{k,l}$  are the skein modules, [T] XII.1.1). In particular, for  $k = l$  we get the homomorphism  $E_k \rightarrow \text{End}(W^{\otimes k})$  where  $E_k = E_{k,k}$  is the Temperley-Lieb algebra. Let  $f_k \in E_k$  be the Jones-Wenzl projectors (see [T] XII.4.1). Set  $a = \sqrt{-q}$  and recall that Turaev defined the category  $\mathcal{V}(a)$  (see [T] XII.6) objects of which are sequences  $(j_1, j_2, \dots, j_l) \in \mathbb{Z}_{\geq 0}^l$ . Define the functor  $\tilde{F} : \mathcal{V}(a) \rightarrow \mathcal{D}$  by  $\tilde{F}((j_1, j_2, \dots, j_l)) = \text{Im}(f_{j_1} \otimes f_{j_2} \otimes \dots \otimes f_{j_l}) \subset W^{\otimes(j_1+j_2+\dots+j_l)}$  and endow it with the obvious tensor structure; since the morphisms in the category  $\mathcal{V}(a)$  are defined in terms of the skein modules, the functor  $\tilde{F}$  is well defined on morphisms. Let us apply this construction to the case  $\mathcal{D} = \tilde{\mathcal{C}}_q$ ,  $W = V$ ,  $\Phi = \phi$ . We get the functor  $\tilde{F}_q : \mathcal{V}(a) \rightarrow \tilde{\mathcal{C}}_q$ . The calculations in [T] XII.8 show that the functor  $\tilde{F}_q$  is an equivalence of categories. Thus we can set  $F = \tilde{F} \circ \tilde{F}_q^{-1}$  and the Theorem is proved.  $\square$

**Remark 2.2.** (i) One can require from  $\mathcal{D}$  to be only a Karoubian category, that is an additive category where any projector has an image.

(ii) Theorem 2.1 implies immediately that the categories  $\tilde{\mathcal{C}}_q$  and  $\tilde{\mathcal{C}}_{q^{-1}}$  are equivalent. Of course this is well known. Another fact of a similar kind is the following. Let  $\tilde{\mathcal{C}}_q^\pm \subset \tilde{\mathcal{C}}_q$  be the full subcategories with objects which are direct sums of odd/even dimensional simple comodules depending on  $\pm$ . Clearly,  $\tilde{\mathcal{C}}_q = \tilde{\mathcal{C}}_q^+ \oplus \tilde{\mathcal{C}}_q^-$ . Moreover,  $\tilde{\mathcal{C}}_q^+ \otimes \tilde{\mathcal{C}}_q^+ \subset \tilde{\mathcal{C}}_q^+$ ,  $\tilde{\mathcal{C}}_q^+ \otimes \tilde{\mathcal{C}}_q^- \subset \tilde{\mathcal{C}}_q^-$ ,  $\tilde{\mathcal{C}}_q^- \otimes \tilde{\mathcal{C}}_q^+ \subset \tilde{\mathcal{C}}_q^-$ ,  $\tilde{\mathcal{C}}_q^- \otimes \tilde{\mathcal{C}}_q^- \subset \tilde{\mathcal{C}}_q^+$ . In other words, the category  $\tilde{\mathcal{C}}_q$  is  $\mathbb{Z}/2\mathbb{Z}$ -graded. Now one can twist the associativity isomorphism in  $\tilde{\mathcal{C}}_q$  by changing the sign of the associativity isomorphism  $(X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z)$  when  $X, Y, Z \in \tilde{\mathcal{C}}_q^-$ . Let us denote the category twisted in such a way by  $\tilde{\mathcal{C}}_q^{tw}$ . It follows from Theorem 2.1 that the category  $\tilde{\mathcal{C}}_q^{tw}$  is equivalent to  $\tilde{\mathcal{C}}_{-q}$ . This is also well known; moreover both facts above remain true when  $q$  is a root of unity.

(iii) One should be very careful with universal properties of tensor categories: for example the universal category with an object  $X$  such that  $X \otimes X = \mathbf{1}$  does

not exist. We expect that when  $q$  is a root of unity the universal abelian category in a sense of Theorem 2.1 does not exist. In contrast the universal Karoubian category clearly exists (it coincides with Karoubian envelope of the skein category from [T] XII.2) and coincides with the category of tilting modules  $\mathcal{T}_q \subset \tilde{\mathcal{C}}_q$  (this is a consequence of the quantized Schur-Weyl duality, see [D]).

(iv) One can restate Theorem 2.1 in the following way: the tensor functors  $F : \tilde{\mathcal{C}}_q \rightarrow \mathcal{D}$  are in the one to one correspondence with objects  $W \in \mathcal{D}$  together with an isomorphism  $\Phi : W \rightarrow W^*$  such that the composition (2) equals to  $-q - q^{-1}$ .

**2.3. Turaev's construction: the roots of unity case.** In the case when  $q$  is a root of unity of order  $N \geq 3$  the category  $\tilde{\mathcal{C}}_q$  is not semisimple, see [K]. Let  $\mathcal{T}_q$  denote the full additive (nonabelian) subcategory of  $\tilde{\mathcal{C}}_q$  whose objects are direct summands of  $V^{\otimes n}$ ; clearly the subcategory  $\mathcal{T}_q$  is closed under the tensor product (the category  $\mathcal{T}_q$  is the category of tilting modules, see e.g. [BK]). It is well known that the additive subcategory  $\mathcal{I}_q$  of  $\mathcal{T}_q$  generated by indecomposable modules of zero quantum dimension is a tensor ideal and thus the quotient  $\mathcal{C}_q = \mathcal{T}_q/\mathcal{I}_q$  is a well defined semisimple tensor category, see [BK]. The object  $V \in \tilde{\mathcal{C}}_q$  can be considered as an object of  $\mathcal{C}_q$  and for the isomorphism  $\phi : V \rightarrow V^*$  the composition (2) equals to  $-q - q^{-1}$ . But the universal property of the category  $\mathcal{C}_q$  is a little bit more delicate. Let  $\mathcal{D}$  be an abelian monoidal category,  $W \in \mathcal{D}$  be a right rigid object and  $\Phi : W \rightarrow W^*$  be an isomorphism such that the composition (2) equals to  $-q - q^{-1}$ . In the same way as in the discussion of Theorem 2.1 we have homomorphisms  $E_{k,l} \rightarrow \text{Hom}(W^{\otimes k}, W^{\otimes l})$  where  $E_{k,l}$  are the skein modules. Set  $N_* = N$  if  $N$  is odd and  $N_* = N/2$  if  $N$  is even. Recall that the last Jones-Wenzl idempotent which is possible to define is  $f_{N_*-1}$ , see [T] XII.4.3.

**Theorem 2.3.** *Assume that  $q$  is a root of unity of order  $N \geq 3$ . The triple  $(\mathcal{C}_q, V, \phi)$  has the following universal property: let  $\mathcal{D}$  be an abelian monoidal category, let  $W \in \mathcal{D}$  be a right rigid object and  $\Phi : W \rightarrow W^*$  be an isomorphism such that the composition morphism*

$$\mathbf{1} \xrightarrow{\text{coev}_W} W \otimes W^* \xrightarrow{\Phi \otimes \Phi^{-1}} W^* \otimes W \xrightarrow{\text{ev}_W} \mathbf{1} \quad (2)$$

*equals to  $-(q + q^{-1})id_{\mathbf{1}}$ . In addition let us assume that the image of  $f_{N_*-1}$  in  $\text{Hom}(W^{\otimes N_*-1}, W^{\otimes N_*-1})$  is zero. Then there exists a unique tensor functor  $F : \mathcal{C}_q \rightarrow \mathcal{D}$  such that  $F(V) = W$  and  $F(\Phi) = \phi$ .*

*In other words, the tensor functors  $F : \mathcal{C}_q \rightarrow \mathcal{D}$  are in bijection with objects  $W \in \mathcal{D}$  together with an isomorphism  $\Phi : W \rightarrow W^*$  such that the composition (2) equals to  $-q - q^{-1}$  and the image of  $f_{N_*-1}$  in  $\text{Hom}(W^{\otimes N_*-1}, W^{\otimes N_*-1})$  is zero.*

**Remarks on proof.** As it is explained in Remark 2.2 (iii) we have a unique tensor functor  $F : \mathcal{T}_q \rightarrow \mathcal{D}$  such that  $F(V) = W$  and  $F(\Phi) = \phi$ . So one just needs to check that this functor maps  $\mathcal{I}_q$  to zero. But it is well known (see e.g. [BK]) that the tensor ideal  $\mathcal{I}_q$  is generated by  $\text{Im}(f_{N_*-1})$  in a sense that any object of  $\mathcal{I}_q$  is isomorphic to a direct summand of  $\text{Im}(f_{N_*-1}) \otimes T$  where  $T \in \mathcal{T}_q$ .  $\square$

**Remark 2.4.** In [T] only the case of even  $N$  is considered. We note that the construction of [T] works without any change for odd  $N$  as well. The only difference is that the resulting semisimple category is not modular.

2.4. **Main equation.** Define

$$\mathcal{C}_q = \begin{cases} \tilde{\mathcal{C}}_q & \text{if } q \text{ is not a root of unity or } q = \pm 1; \\ \mathcal{T}_q/\mathcal{I}_q & \text{if } q \text{ is a root of unity, } q \neq \pm 1. \end{cases}$$

The aim of this note is to classify the semisimple module categories with finitely many simple objects over the category  $\mathcal{C}_q$ . Here is our main result:

**Theorem 2.5.** (i) *Assume that  $q$  is not a root of unity or  $q = \pm 1$ . The semisimple module categories with finitely many simple objects over the category  $\mathcal{C}_q$  are classified by the following data:*

- 1) *A finite set  $I$ ;*
- 2) *A collection of finite dimensional vector spaces  $V_{ij}$ ,  $i, j \in I$ ;*
- 3) *A collection of nondegenerate bilinear forms  $E_{ij} : V_{ij} \otimes V_{ji} \rightarrow \mathbb{C}$ ,*  
*subject to the following condition: for each  $i \in I$  we have*

$$\sum_{j \in I} \text{Tr}(E_{ij}(E_{ji}^t)^{-1}) = -q - q^{-1}. \quad (3)$$

*Proof.* Let  $\mathcal{M}$  be a semisimple module category over  $\mathcal{C}_q$  with finitely many simple objects. Let  $I$  be the set of the isomorphism classes of simple objects in  $\mathcal{M}$ . The structure of module category on  $\mathcal{M}$  is the same as the tensor functor  $F : \mathcal{C}_q \rightarrow \text{Fun}(\mathcal{M}, \mathcal{M})$  where  $\text{Fun}(\mathcal{M}, \mathcal{M})$  is the category of additive functors from  $\mathcal{M}$  to itself, see [O]. Recall that the category  $\text{Fun}(\mathcal{M}, \mathcal{M})$  is identified with the category of  $I \times I$ -graded vector spaces with obvious ‘‘matrix’’ tensor product. By Remark 2.2 (iv) the functors  $F : \mathcal{C}_q \rightarrow \text{Fun}(\mathcal{M}, \mathcal{M})$  are bijective to the objects  $\bar{V} = (V_{ij}) \in \text{Fun}(\mathcal{M}, \mathcal{M})$  together with an isomorphism  $\Phi : \bar{V} \rightarrow \bar{V}^* = (V_{ji}^*)$  (equivalently, this is a collection of nondegenerate bilinear forms  $E_{ij} : V_{ij} \otimes V_{ji} \rightarrow \mathbb{C}$ ) such that the morphism (2) equals to  $-q - q^{-1}$ . It is obvious that the last condition is equivalent to the condition (3). The theorem is proved.  $\square$

### 3. SOLUTIONS OF THE MAIN EQUATION

3.1. **Deformations.** Let us fix a finite set  $I$ , the numbers  $a_{ij} = \dim V_{ij} = \dim V_{ji}$  and try to analyze the corresponding solutions of the main equation. It is convenient to represent these data as a graph  $\Gamma = (I, \{a_{ij}\})$  with the set of vertices  $I$  and  $a_{ij}$  edges joining the vertices  $i$  and  $j$  (the matrix  $A = (a_{ij})$  is the adjacency matrix of this graph). We are going to classify the graphs with respect to the deformation behavior of solutions of the main equation. If we fix the vector spaces  $V_{ij}$  of dimensions  $a_{ij}$ , the set of solutions of equation (3) is clearly an affine algebraic variety  $\mathfrak{M}$ . The group  $G = \prod_{i,j} GL(V_{ij})$  acts naturally on  $\mathfrak{M}$ . Since we are interested in the solutions of the main equation only up to isomorphism we define the set of solutions of the main equation to be the set  $\mathfrak{M}/G$  of orbits of  $G$  on  $\mathfrak{M}$ . In general  $\mathfrak{M}/G$  has no structure of an algebraic variety; so let  $\mathfrak{M}/\!/G$  denote the quotient in the sense of the invariant theory, that is  $\mathfrak{M}/\!/G$  is the set of closed  $G$ -orbits on  $\mathfrak{M}$ . Now  $\mathfrak{M}/\!/G$  has a structure of an algebraic variety; we will see that the natural map  $\mathfrak{M}/G \rightarrow \mathfrak{M}/\!/G$  is finite to one and is one to one on an open nonempty subset of  $\mathfrak{M}/\!/G$ . Thus we define the dimension of the set of solutions of the main equation to be equal to the dimension of the variety  $\mathfrak{M}/\!/G$ . In such situation we will say that  $\mathfrak{M}/G$  is a moduli space (even if it is not an algebraic variety).

**Definition 3.1.** (i) We will say that a graph is super-rigid if the main equation (3) admits only finitely many solutions for finitely many values of  $q$  and no solutions for other values of  $q$ .

(ii) We will say that a graph is rigid if the main equation (3) admits only finitely many solutions for all but finitely many values of  $q$ .

(iii) We will say that a graph is non-rigid if it is not rigid.

**Remark 3.2.** One says that a graph is strictly rigid if the main equation admits only finitely many solutions for all values of  $q$ . We will see later that the graph  is rigid. On the other hand it is easy to see that it is not strictly rigid: for  $q + q^{-1} = 1$  it admits infinitely many solutions of the main equation.

If a graph  $\Gamma$  is a disjoint union of two subgraphs, the coresponding module category over  $\mathcal{C}_q$  is clearly a direct sum of module subcategories corresponding to the subgraphs. Thus from now on we will study only connected graphs. For a graph  $\Gamma$  we define its *underlying simply laced graph* as a graph  $\bar{\Gamma}$  with the same set of vertices and the vertices  $i \neq j$  are joined by exactly one edge if  $v_{ij} \neq 0$  and are not joined otherwise (in particular  $\bar{\Gamma}$  has no self-loops).

For a graph  $\Gamma = (I, \{a_{ij}\})$  we define the generalized number of cycles  $L(\Gamma)$  by formula

$$L(\Gamma) = \frac{1}{2} \sum_{i \neq j} a_{ij} + \sum_i \left[ \frac{a_{ii}}{2} \right] - |I| + 1$$

where  $[\cdot]$  denotes the integer part. Note that in a case when  $\Gamma$  has no self-loops  $L(\Gamma)$  is just the number of loops in  $\Gamma$ . We will see later that the number  $L(\Gamma)$  is the expected dimension (that is, the difference of the number of variables and the number of equations) of the set of solutions of the main equation. Moreover, we will see that  $L(\Gamma)$  coincides with (properly understood) dimension of the set of solutions of the main equation.

**Definition 3.3.** (i) A connected graph is called a generalized tree if  $L(\Gamma) = 0$ .

(ii) A connected graph is called a 1-loop graph if  $L(\Gamma) = 1$ .

**Remark 3.4.** (i) A connected graph is a generalized tree if  $a_{ij} \leq 1$  and its underlying simply laced graph is a tree (note that the possibility  $a_{ii} \neq 0$  is allowed).

(ii) A connected graph is a 1-loop graph if either  $a_{ij} \leq 1$  and its underlying simply laced graph has exactly  $|I|$  edges or its underlying simply laced graph is a tree,  $a_{ij} \leq 3$ ,  $a_{ij} \geq 2$  for exactly one pair of vertices  $i, j$  and  $a_{ij} = 3$  implies  $i = j$ .

**Theorem 3.5.** (i) For any graph there exists a solution of the main equation with some  $q \neq \pm i$ .

(ii) A connected graph is super-rigid iff it is a generalized tree.

(iii) A connected graph is rigid but not super-rigid iff it is a 1-loop graph.

*Proof.* 1) A quadruple  $(V_{ij}, V_{ji}, E_{ij}, E_{ji})$  consisting of two vector spaces  $V_{ij}, V_{ji}$  and two nondegenerate bilinear forms  $E_{ij} : V_{ij} \otimes V_{ji} \rightarrow \mathbb{C}$  and  $E_{ji} : V_{ji} \otimes V_{ij} \rightarrow \mathbb{C}$  is isomorphic to the quadruple  $(V_{ij}, V_{ij}^*, \langle \cdot, \cdot \rangle, \langle S \cdot, \cdot \rangle)$  where  $\langle \cdot, \cdot \rangle : V_{ij} \otimes V_{ij}^* \rightarrow \mathbb{C}$  is the canonical pairing and  $S : V_{ij} \rightarrow V_{ij}$  is an invertible linear operator; two such quadruples are isomorphic if and only if the corresponding operators  $S$  have the same Jordan form. Thus the moduli space  $\mathcal{Q}(a_{ij})$  of such quadruples with  $\dim(V_{ij}) = \dim(V_{ji}) = a_{ij}$  has dimension  $a_{ij}$ . The image of the map  $\mathcal{Q}(a_{ij}) \rightarrow \mathbb{A}^2$ ,  $x = \text{Tr}(E_{ij}(E_{ji}^t)^{-1})$ ,  $y = \text{Tr}(E_{ji}(E_{ij}^t)^{-1})$  depends on  $a_{ij}$ : if  $a_{ij} = 1$  this is the hyperbola  $xy = 1$ ; if  $a_{ij} = 2$  this is  $(\mathbb{A}^2 - \{xy = 0\}) \cup (0, 0)$ ; if  $a_{ij} \geq 3$  this is  $\mathbb{A}^2$ .

2) Recall here the classification of nondegenerate bilinear forms, see [B], 5.6: any pair  $(V, E)$  consisting of a vector space  $V$  and nondegenerate bilinear form  $E : V \otimes V \rightarrow \mathbb{C}$  is up to isomorphism uniquely determined by the operator  $S_E = E(E^t)^{-1} : V^* \rightarrow V^*$ ; the operator  $S_E$  is conjugated to  $S_E^{-1}$  and moreover the number of Jordan cells of size  $k$  with eigenvalue  $(-1)^k$  is even. Thus the moduli space  $\tilde{\mathcal{Q}}(a)$  of such pairs with  $\dim(V) = a$  has dimension  $[a/2]$  (in the same sense as before); the image of the map  $\tilde{\mathcal{Q}}(a) \rightarrow \mathbb{A}^1$ ,  $(V, E) \mapsto \text{Tr}(E(E^t)^{-1})$  is just the point 1 for  $a = 1$  and the entire  $\mathbb{A}^1$  for  $a \geq 2$ .

Thus we see that  $L(\Gamma)$  is really the expected dimension of the set of solutions of the main equation. It is clear that the actual dimension of the set of solutions of the main equation is greater or equal to  $L(\Gamma)$  if this set is nonempty..

Now let us show that for any choice of the graph  $\Gamma$  there exists a solution of the main equation. Let  $(r_i)_{i \in I}$  be an eigenvector of the matrix  $A = (a_{ij})$  with eigenvalue  $\lambda$  and such that  $\prod_{i \in I} r_i \neq 0$  (such eigenvector exists, for example one can take the Frobenius-Perron eigenvector). Now choose bilinear forms  $E_{ij}$  in such a way that  $\text{Tr}(E_{ij}(E_{ji}^t)^{-1}) = a_{ij}r_j/r_i$  (this is possible in view of the remarks above). It is clear that in this way we get a solution of the main equation with  $\lambda = -q - q^{-1}$ . If  $\lambda$  is the Frobenius-Perron eigenvalue we have  $\lambda > 0$  and thus  $q \neq \pm i$ . Thus (i) is proved.

Now assume that the graph  $\Gamma$  is not a generalized tree. Thus either  $a_{i_0 j_0} \geq 2$  for some  $i_0, j_0 \in I$ , or our graph contains a cycle of length  $M \geq 3$ .

Case 1:  $i_0 = j_0$ . Consider the matrix  $\tilde{A}(u) = (\tilde{a}_{ij})$  where  $\tilde{a}_{ij} = a_{ij}$  except  $\tilde{a}_{i_0 i_0} = u \in \mathbb{C}$ . For real positive  $u$  the Frobenius-Perron eigenvalue of the matrix  $\tilde{A}(u)$  depends nontrivially on  $u$  since  $\text{Tr}(\tilde{A}(u)) = u + \text{Tr}(A) - a_{i_0 i_0}$ . Thus for generic  $u$  the matrix  $\tilde{A}(u)$  has an eigenvector  $(r_i(u))_{i \in I}$  with  $\prod_{i \in I} r_i \neq 0$  and with an eigenvalue  $\lambda(u)$  depending nontrivially on  $u$ . Thus  $\lambda(u)$  takes all values from  $\mathbb{C}$  except finitely many. Now a choice of  $E_{ij}$  such that  $\text{Tr}(E_{ij}(E_{ji}^t)^{-1}) = \tilde{a}_{ij}(u)r_j(u)/r_i(u)$  (this choice is possible by 1) and 2) above) gives a solution of the main equation with  $-q - q^{-1} = \lambda(u)$ . Thus our graph is not super-rigid.

Case 2:  $i_0 \neq j_0$ . In this case consider the matrix  $\tilde{A}(u) = (\tilde{a}_{ij})$  where  $\tilde{a}_{ij} = a_{ij}$  except  $\tilde{a}_{i_0 j_0} = u \in \mathbb{C}$ . Since  $\text{Tr}(\tilde{A}(u)^2)$  depends on  $u$  nontrivially the same arguments as above show that our graph is not super-rigid.

Case 3: the graph has a cycle of length  $M \geq 3$ . Let  $(i_0, j_0)$  be an edge from the cycle. Consider the matrix  $\tilde{A}(u) = (\tilde{A}_{ij})$  where  $\tilde{A}_{ij} = A_{ij}$  except  $\tilde{a}_{i_0 j_0} = u \in \mathbb{C}$  and  $\tilde{a}_{j_0 i_0} = u^{-1}$ . Now  $\text{Tr}(\tilde{A}(u)^M)$  depends on  $u$  nontrivially and our graph is not super-rigid.

Now we are going to prove that a generalized tree is super-rigid.

**Definition 3.6.** An eigenvalue  $\lambda$  of the matrix  $A = (a_{ij})$  is called nondegenerate if there exists a  $\lambda$ -eigenvector  $(r_i)_{i \in I}$  such that  $\prod_{i \in I} r_i \neq 0$ .

The following lemma is well known in graph theory, see [F]. We give a proof for the reader's convenience.

**Lemma 3.7.** (i) For any matrix  $A$  with nonnegative integer entries there exists a nondegenerate eigenvalue.

(ii) If  $A$  corresponds to a generalized tree then a nondegenerate eigenvalue has multiplicity 1.

*Proof.* (i) The Frobenius-Perron eigenvalue (and its Galois conjugates) is always nondegenerate.

(ii) Let  $\lambda$  be an eigenvalue of the matrix  $A$ . We are going to prove that an  $\lambda$ -eigenvector  $(r_i)_{i \in I}$  with  $\prod_{i \in I} r_i \neq 0$  is unique up to proportionality if it exists. This would imply the statement of Lemma since a small perturbation preserves the property  $\prod_{i \in I} r_i \neq 0$ .

The vector  $(r_i)_{i \in I}$  satisfies

$$\sum_{i \neq j, a_{ij}=1} r_i = \begin{cases} \lambda r_j & \text{if } a_{jj} = 0, \\ (\lambda - 1)r_j & \text{if } a_{jj} = 1, \end{cases}$$

(the sum is over all edges of the underlying simply laced graph with vertex  $j$ ). Let us introduce new variables parametrized by the oriented edges of the underlying simply laced graph,  $y_{ij} = r_i/r_j$ . These variables satisfy

$$y_{ij}y_{ji} = 1, \quad \sum_{i \neq j, a_{ij}=1} y_{ij} = \begin{cases} \lambda & \text{if } a_{jj} = 0, \\ \lambda - 1 & \text{if } a_{jj} = 1. \end{cases} \quad (4)$$

Now the result is a consequence of the following

**Sublemma.** For any choice of  $(\lambda_j)_{j \in I}$  the system of equations

$$y_{ij}y_{ji} = 1, \quad \sum_{i \neq j, a_{ij}=1} y_{ij} = \lambda_j \quad (5)$$

has at most one solution.

**Proof of Sublemma.** The proof is by induction in  $|I|$ . Choose a vertex  $j_0$  of valency 1 of the underlying simply laced graph (which is a tree). Then there is only one variable  $y_{ij_0}$  and it is uniquely defined from the equation  $y_{ij_0} = \sum_{i \neq j_0, a_{ij_0}=1} y_{ij_0} = \lambda_{j_0}$ . Then the variable  $y_{j_0i} = 1/y_{ij_0}$  is also uniquely defined and all other variables satisfy the system of equations of the same form with smaller  $|I|$ . The Sublemma and the Lemma are proved.  $\square$

Observe that in the case of a generalized tree the main equation has exactly the form of system (4). Thus it is obvious that the only possible values of  $\lambda$  are the nondegenerate eigenvalues of  $A$ . So the Sublemma implies that a generalized tree is super-rigid. Thus (ii) is proved.

Now we claim that for any graph  $\Gamma$  the dimension of the space of solutions of the main equation is exactly  $L(\Gamma)$ . Indeed, let us choose a spanning tree  $T$  of the underlying simply laced graph  $\bar{\Gamma}$ . Now let us choose any values of parameters attached to all edges not belonging to  $T$ ; in particular for any edge  $ij$  from  $T$  choose any values of  $a_{ij} - 1$  eigenvalues of the matrix  $S$  (see 1) above). Thus we have chosen  $L(\Gamma)$  parameters. Now the main equation reduces to the system of the shape (5) for the rest of parameters (we have one parameter for each edge of the tree  $T$ ). Now the Sublemma implies that we have only finitely many solutions for these parameters. Henceforth we see that the expected dimension  $L(\Gamma)$  coincides with the actual dimension (understood as it is explained above) of the set of solutions of the main equation.

Now it is clear that if a graph is a 1-loop graph if and only if it is rigid (indeed, the set of solutions of the main equation has dimension 1 and it maps dominantly under the projection to the variable  $q$ ). The Theorem is proved.  $\square$

**Corollary 3.8.** *For any value of  $|I|$  there are only finitely many rigid (and hence super-rigid) graphs.*

Recall that the ultraspherical polynomials  $P_n(x)$  are defined recursively by

$$P_1(x) = 1, P_2(x) = x, P_{n+2}(x) = xP_{n+1}(x) - P_n(x), n \geq 1.$$

It is a classical fact that the classes of simple objects in  $\text{Gr}(\mathcal{C}_q)$  are given by  $P_n([V])$ , see e.g. [K].

**Corollary 3.9.** *Let  $A$  be an indecomposable symmetric matrix with nonnegative entries. Then either  $P_n(A) = 0$  for some  $n$  (all such matrices are explicitly known and are classified by ADET graphs, see below) or  $P_n(A)$  has nonnegative entries for all  $n$ .*

*Proof.* Let  $\lambda$  be the Frobenius-Perron eigenvalue of  $A$ . All indecomposable matrices with  $\lambda < 2$  are classified and it is well known that for such matrices  $P_n(A) = 0$  for some  $n$ , see e. g. [EK]. Hence we can assume that  $\lambda \geq 2$ . Then the construction from the proof of Theorem 3.5 gives us a module category over  $\mathcal{C}_q$  with  $q + q^{-1} = \lambda$  (and thus  $q$  is not a root of unity) where the object  $V$  is represented by the vector space valued matrix of dimension  $A$ . Since  $P_n(A)$  gives an action of some object in  $\mathcal{C}_q$ , it is nonnegative.  $\square$

Consider the case  $q = 1$ . In this case  $\mathcal{C}_1 \cong \text{Rep}(SL(2))$ . Any finite subgroup  $G \subset SL(2)$  gives rise to a module category  $\text{Rep}(G)$  over  $\text{Rep}(SL(2))$ . As it is known from the McKay correspondence, the corresponding graph is then an affine ADE Dynkin diagram. Observe that the graphs of type  $D_n, \tilde{E}_n$  are super-rigid while the graph of type  $\tilde{A}_n$  is just rigid. This explains the fact that among finite subgroups of  $SL(2)$  only the cyclic subgroups corresponding to  $\tilde{A}_n$  admit a continuous deformation in the “quantum” direction.

**3.2. Examples.** In this section we will assume that  $q$  is not a root of unity. Recall that the module categories over  $\mathcal{C}$  with one simple object are the same as fiber functors (= tensor functors  $\mathcal{C} \rightarrow \text{Vec}$ ). We see from Theorem 2.5 that the fiber functors on  $\mathcal{C}_q$  are classified by a vector space  $V$  and a bilinear form  $E : V \otimes V \rightarrow \mathbb{C}$  such that  $\text{Tr}(E(E^t)^{-1}) = -q - q^{-1}$ . This is exactly the result of J. Bichon, see [Bi] who classified all Hopf algebras  $H$  such that the category of comodules over  $H$  is tensor equivalent to  $\mathcal{C}_q$ . Thus our Theorem 2.5 can be considered as a generalization of Bichon’s result: we classify all weak Hopf algebras  $H$  such that the category of comodules over  $H$  is tensor equivalent to  $\mathcal{C}_q$ .

Observe that a graph with one vertex is rigid iff  $\dim(V) \leq 3$  and is super-rigid iff  $\dim(V) = 1$  (the last case gives  $q$  which is a primitive root of unity of order 3). Here is a list of rigid graphs with  $|I| = 2$  (we wrote possible values of  $q + q^{-1}$  over the super-rigid graphs):



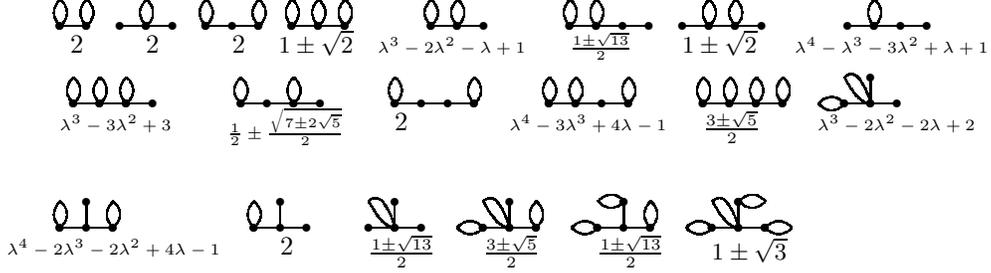
Now we are going to discuss the case of super-rigid graphs. Let  $\Gamma = (I, \{a_{ij}\})$  be a generalized tree. We have the following consequence of Lemma 3.7:

**Proposition 3.10.** *A solution of the main equation (3) for a generalized tree exists if and only if  $-q - q^{-1}$  is a nondegenerate eigenvalue of  $A = (a_{ij})$ . In such a case the solution is unique.*

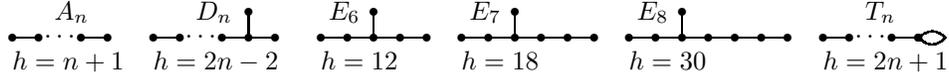
Observe that since an eigenvalue of symmetric matrix is real we have

**Corollary 3.11.** *If the category  $\mathcal{C}_q$  has a module category corresponding to a super-rigid graph then either  $|q| = 1$  or  $q$  is real.*

We present here a few examples. Here is a list of generalized trees with  $\leq 4$  vertices; under any graph we wrote possible values of  $\lambda = -q - q^{-1}$  or an algebraic equation for  $\lambda$ ; we omitted all values of  $q$  being a root of unity (thus some graphs are omitted too; see the next subsection for them).



**3.3. The roots of unity case.** In this case we recover the Ocneanu-Kirillov-Ostrik classification of the module categories over  $\mathcal{C}_q$  (the quantum “McKay correspondence”), see [Oc], [KO], [O]. Let  $q$  be a root of unity of order  $N \geq 3$ . Recall here that the irreducible based modules over the based ring  $\text{Gr}(\mathcal{C}_q)$  are classified by the ADET Dynkin diagrams with the Coxeter number  $N_*$ , see [DZ], [EK]. In the pictures below the subscript is the number of vertices and  $h$  is the Coxeter number:



**Theorem 3.12.** *Let  $q$  be a primitive root of unity of order  $N \geq 3$ .*

(i) *Assume that  $N$  is even. The indecomposable module categories over the category  $\mathcal{C}_q$  are classified by the ADE Dynkin diagrams with the Coxeter number  $N_* = N/2$ .*

(ii) *Assume that  $N$  is odd. The indecomposable module categories over the category  $\mathcal{C}_q$  are classified by the ADET Dynkin diagrams with the Coxeter number  $N_* = N$ .*

*Proof.* It is clear that any module category  $\mathcal{M}$  over  $\mathcal{C}_q$  gives rise to a based module  $\text{Gr}(\mathcal{M})$  over  $\text{Gr}(\mathcal{C}_q)$ . Such based modules were classified in [DZ], [EK] and the answer is given precisely by ADET Dynkin diagrams with the Coxeter number  $N_*$ . Conversely, we know that any generalized tree with nondegenerate eigenvalue  $q + q^{-1}$  gives rise to a unique module category over  $\mathcal{T}_q$ . It is well known that the class of the object  $\text{Im}(f_{N_*-1})$  in  $\text{Gr}(\tilde{\mathcal{C}}_q)$  is given by  $P_{N_*}([V])$ , see e.g. [BK] (recall that  $P_{N_*}$  is an ultraspherical polynomial). On the other hand in  $\text{Gr}(\mathcal{C}_q)$  we have the relation  $P_{N_*}([V]) = 0$ , see *loc. cit.* Thus we can apply Theorem 2.3. Observe that  $-q - q^{-1}$  is a nondegenerate eigenvalue of the adjacency matrix of the corresponding graph  $\Gamma$  in all cases except when  $N$  is even and  $\Gamma = T_n$  (actually in all cases  $-q - q^{-1}$  is Galois conjugate to the Frobenius-Perron eigenvalue). Thus by Proposition 3.10 we have a unique solution of the main equation. The Theorem is proved.  $\square$

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